MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 21

Properties of differentiable functions

Derivative of compositions
(~~Produ~~ chain rule)

Relative maximum/minimum

1.

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Chain - rule

Suppose that g has a derivative at x_0 and f has a derivative at $g(x_0)$. Then $f \circ g$ has a derivative at x_0

$$(f \circ g)'(x_0) = f'(g(x_0))\, g'(x_0)$$

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$$(f\circ g)'(x_0) = \lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{f(\underbrace{g(x)}_{}) - f(\underbrace{g(x_0)}_{})}{g(x) - g(x_0)} \; \frac{g(x) - g(x_0)}{x - x_0}$$

$$= f'(g(x_0)) \; g'(x_0) .$$

$g(x) \neq g(x_0)$

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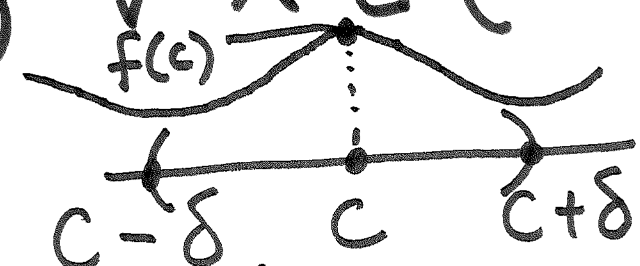
Definition

$f : D \longrightarrow \mathbb{R}$. f has a relative maximum at $c \in D$
(local)

if $\exists\ \delta > 0$ such that

nbd of c

$$f(x) \leq f(c)\ ,\ \forall x \in (c-\delta, c+\delta)$$



relative minimum at c

if $f(x) \geq f(c)\ ,\ \forall x \in (c-\delta, c+\delta)$

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$D = [a, e)$

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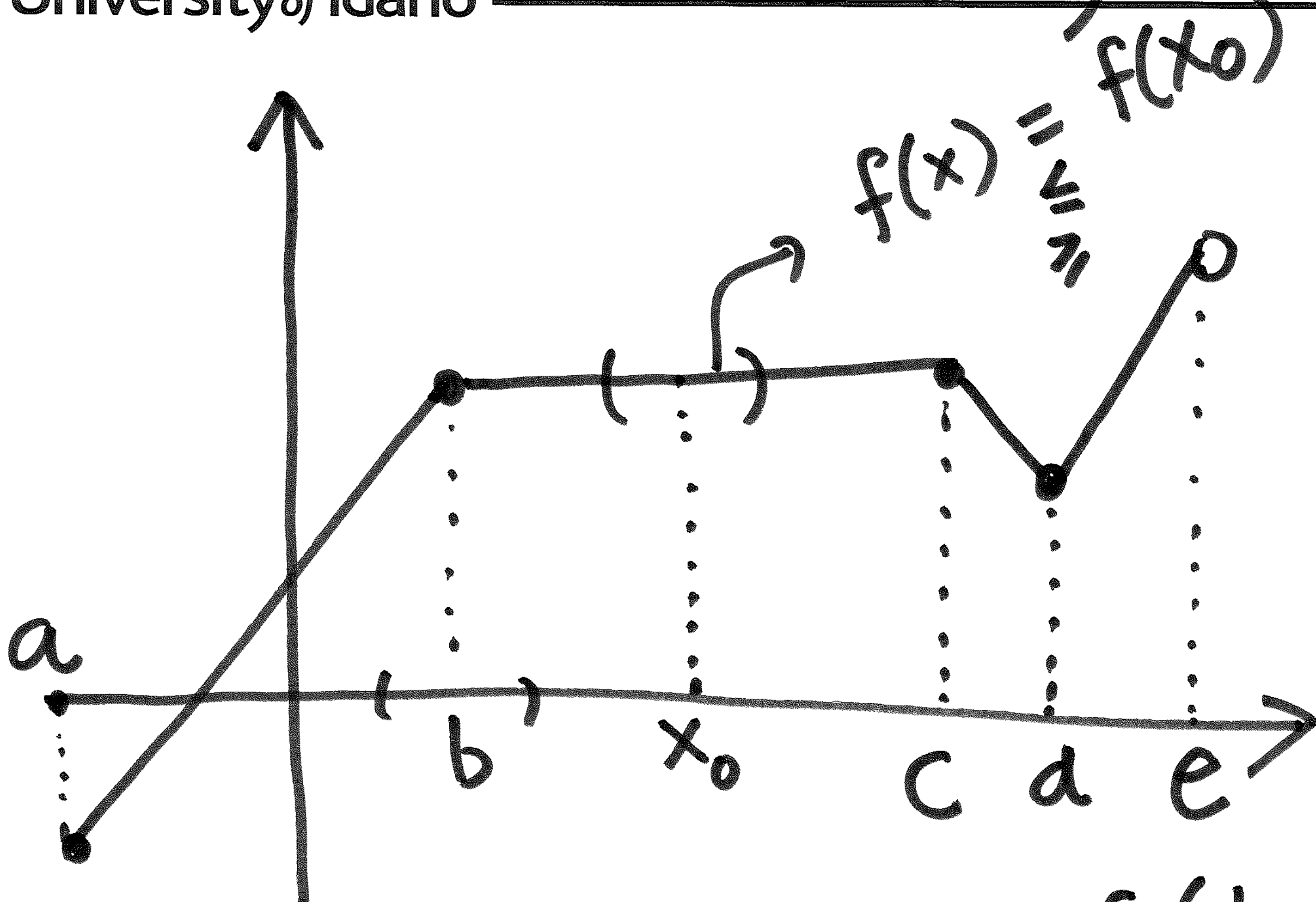

a – local min

b – local max

$x_0 \in (b, c)$: local max or local min

d – local min

e – neither max/min

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Theorem

Suppose that $f : D \to \mathbb{R}$ has a relative $\underbrace{\text{max/min}}_{\text{extremum}}$ at $x_0 \in (a,b) \subseteq D$. If f is differentiable at x_0 then

$$f'(x_0) = 0.$$

Proof (only for max, similar for min)

Let f have a relative $\underline{max}$ at x_0.

Then $\exists\ \delta > 0$ s.t.

$f(x) \leq f(x_0)$ for $x \in (x_0-\delta, x_0+\delta)$

For any h s.t. $-\delta < h < \delta$
$|h| < \delta$

$f(x_0+h) - f(x_0) \leq 0$.

$(x_0 + h \in (x_0-\delta, x_0+\delta)$ &
$f(x_0)$ is a relative max$)$

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$$\frac{f(x_0+h)-f(x_0)}{h} \leq 0 \text{ if } h>0$$

$$\geq 0 \text{ if } h<0$$

$$\lim_{\substack{h\to 0\\ h>0}} \frac{f(x_0+h)-f(x_0)}{h} \leq 0$$

$$\lim_{\substack{h\to 0\\ h<0}} \frac{f(x_0+h)-f(x_0)}{h} \geq 0$$

Since f is differentiable @ x_0 both limits must exist and be equal and this happens only if

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = 0$$

or, $f'(x_0) = 0$ $\square$

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Find an example of a function that has a relative extremum at $c \in (a, b)$ for which $f'(c) \neq 0$.

[Find a function such that $f(c)$ is a relative max/min but $f'(c)$ does not exist].

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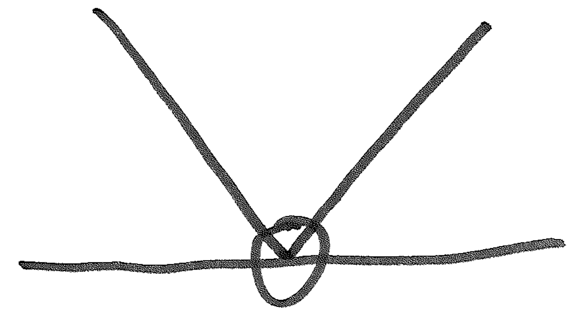
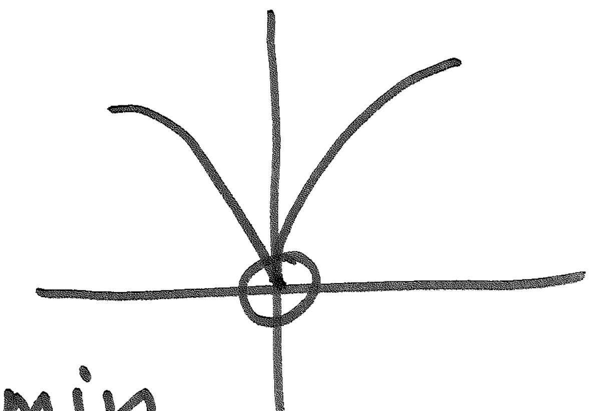
1) $f(x) = |x|$

local min @ $c=0$.



2) $f(x) = x^{2/3}$ defined $(-2, 2)$

$f'(x) = \frac{2}{3}\frac{1}{x^{1/3}}$



$c = 0$ is a local min

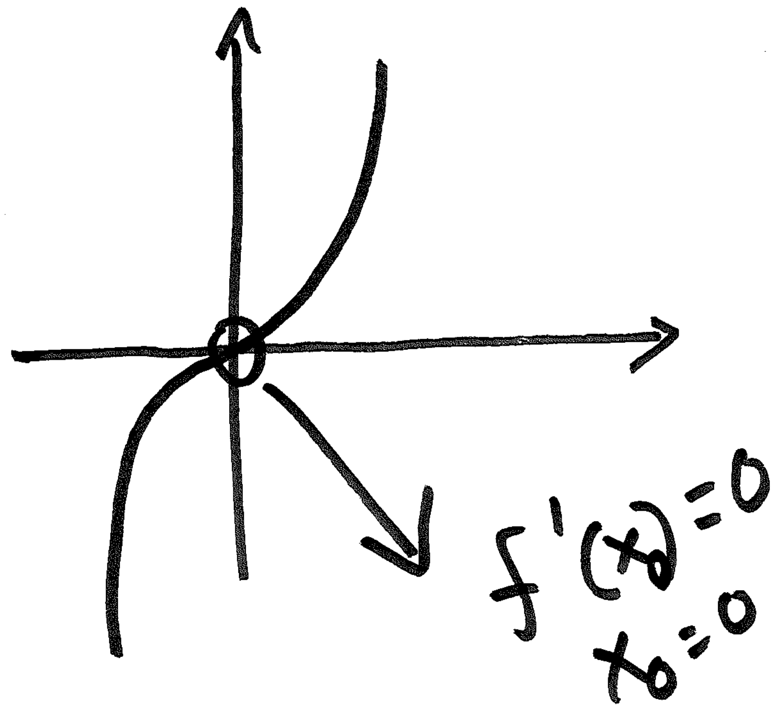
But $f'(0)$ does not exist.

Converse of the theorem:

$f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0$



$x_0 = 0$ is neither a local max not a local min. even though $f'(x_0) = 0$

$\Rightarrow$ Converse of the theorem does not hold.