MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 22

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Today

Differentiation of the inverse function

Rolle's mean value thm

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$f : D \longrightarrow \mathbb{R}$

$$f : D \longrightarrow f(D) \subseteq \mathbb{R}$$
$$x \longmapsto y$$

f is one-to-one

$$f^{-1} : f(D) \longrightarrow D$$
$$y \longmapsto x$$

For a given $y \in f(D)$

$\exists$ only one x s.t. $f(x) = y$.

x is defined to be $f^{-1}(y)$.

$$f^{-1}(y) = x$$

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$f(x) = x^2$ Let $f(x) = y$.

Solve for x.

$$y = x^2$$

$$\Rightarrow x = y^{1/2}$$

$$\Rightarrow f^{-1}(y) = \sqrt{y}$$

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$$f(x) = e^x$$

Let $y = e^x$.

$$\ln y = x$$

$$\Rightarrow f^{-1}(y) = \ln y.$$

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$$f^{-1}(f(x)) = x$$

$$f(f^{-1}(y)) = y$$

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Theorem

$f : D \longrightarrow \mathbb{R}$, $f'(x) \neq 0$, $\forall x \in D$.

D $\downarrow$ interval

Then

(a) f is one-to-one

(b) f^{-1} is Continuous & differentiable

(c) $$(f^{-1})'(y) = \frac{1}{f'(x)}$$ where $y = f(x)$

$x = f^{-1}(y)$

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(a) $D = [a, b]$. $f'(x) \neq 0$

for any $x \in D \Rightarrow f'(x) \neq 0$

for $x \in (a, b) \Rightarrow f$ has

no relative max/min in (a, b).

$\Rightarrow f$ is a strictly monotone function.

$\Rightarrow f$ is one-to-one.

7 $y = f(x)$; $x = f^{-1}(y)$

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b) - Skip

c) $x = f^{-1}(f(x))$

Differentiate both sides w.r.t. x:

$$1 = (f^{-1})'(f(x))\, f'(x)$$

$$\Rightarrow (f^{-1})'(f(x)) = \frac{1}{f'(x)} \quad /\!/$$

$$\Rightarrow (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

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$$f(x) = x^2 \qquad x^2 = y$$

$$\sqrt{y} = x$$

$$f^{-1}(y) = \sqrt{y}$$

$$(f^{-1})'(y) = \frac{1}{2\sqrt{y}} = \frac{1}{2x}$$

$$= \frac{1}{f'(x)}$$

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Rolle's Theorem

Suppose

1) f is continuous on $[a,b]$.

2) f is differentiable on (a,b)

3) $f(a) = f(b)$

Then $\exists$ a point $c \in (a,b)$ s.t.

$$f'(c) = 0$$

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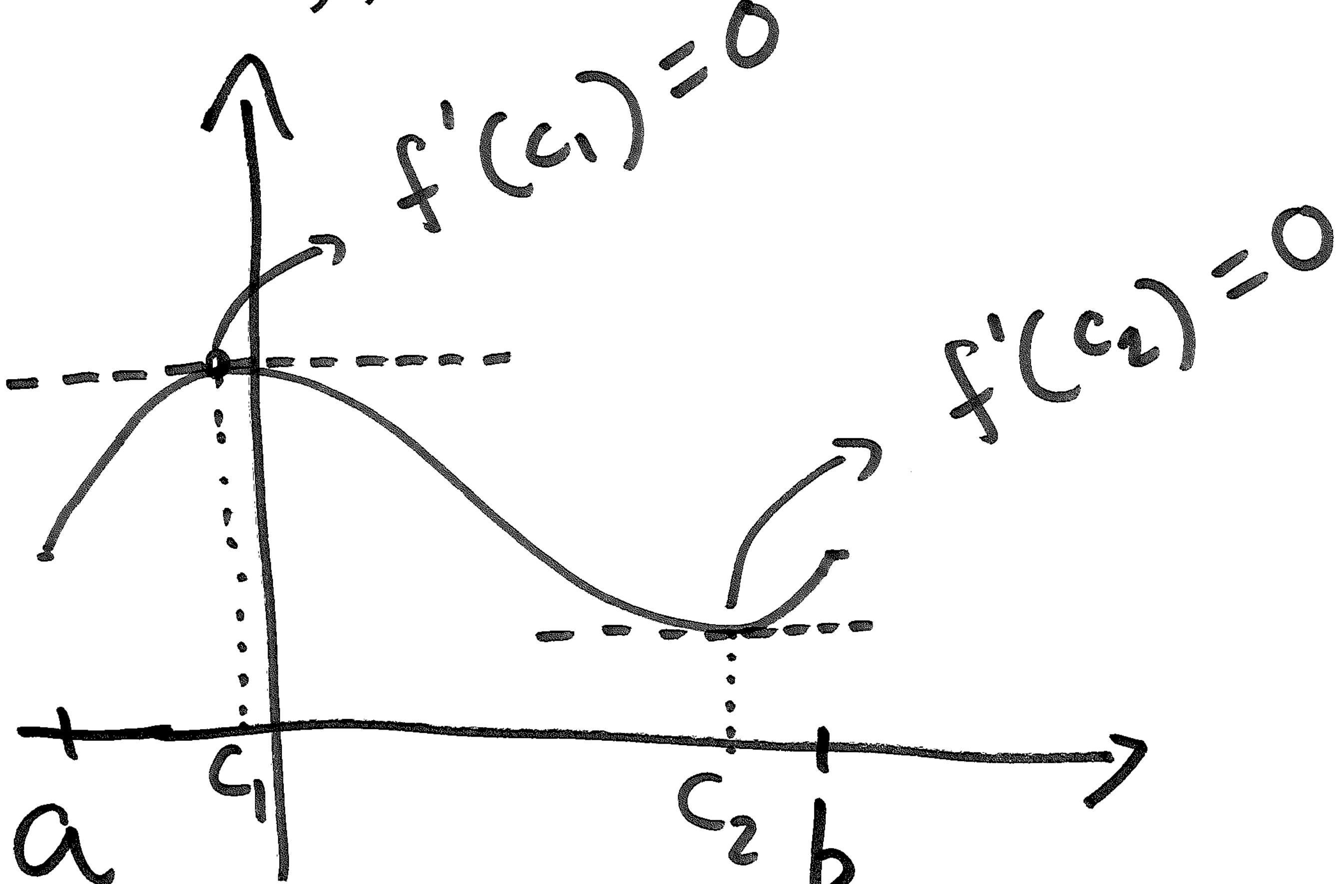

There could be more than one point c where $f'(c)=0$.

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Proof (of Rolle's)

If $f = k$, a constant then $f'(x) = 0$, $\forall x$ & the result holds. Suppose f is not constant. Since f is cont. on a closed & bounded set it must attain a max and a min (by the Extreme Value Thm)

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Let $M = \max(f)$, $m = \min(f)$.

$M \neq m$ since f is not a constant.

Proceed by contradiction:

assume that $f'(x) \neq 0$

$\forall x \in (a,b)$.

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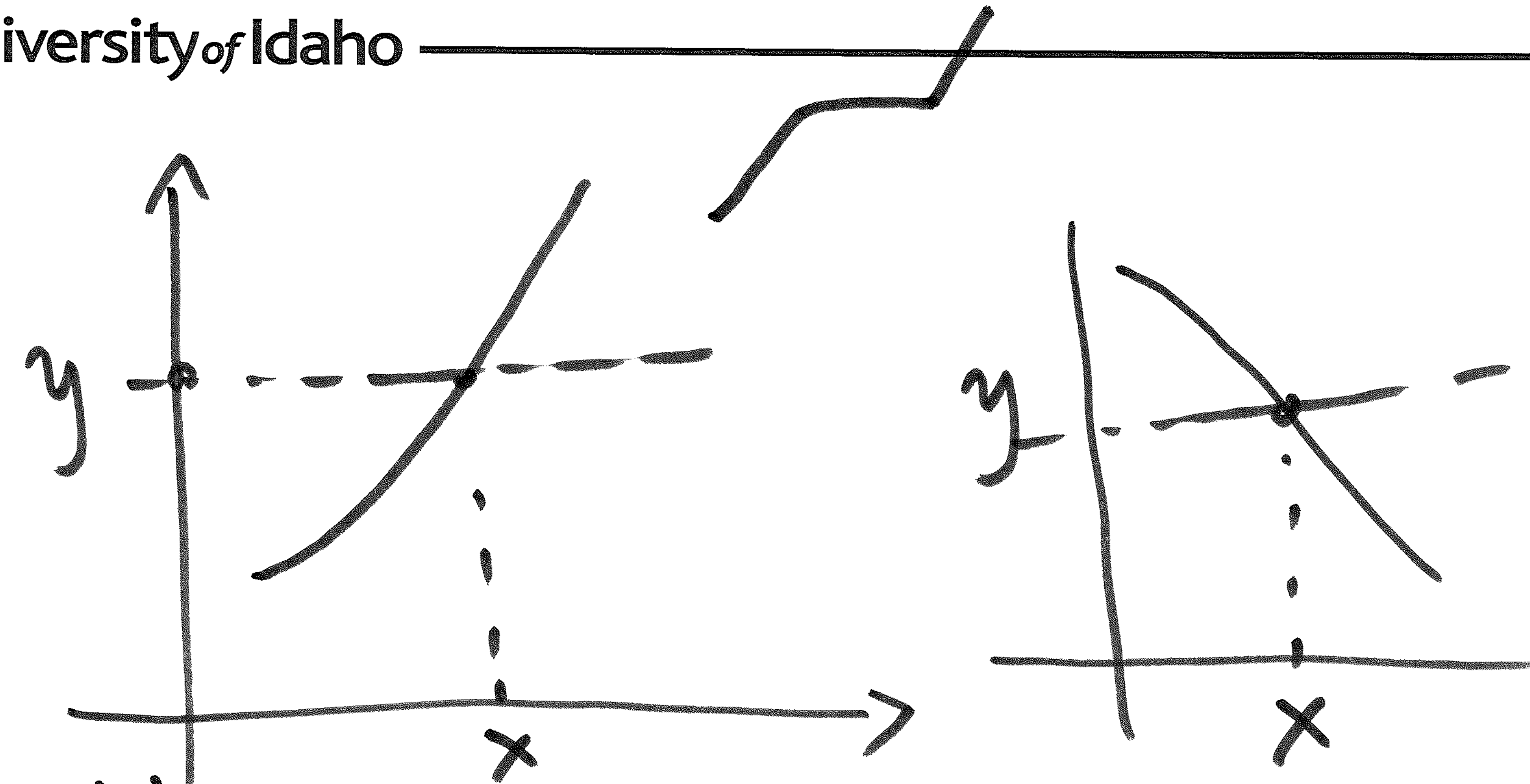
Since f is differentiable, this implies that f cannot attain a max. or a min in (a, b). So the extremum must be at a & b. But $f(a) = f(b)$ and max $\neq$ min $\Rightarrow \exists\, c$ s.t. $f(c) = 0$, $c \in (a, b)$

$\square$



Strictly monotone $\Rightarrow$ one-one.

$$x_1 \overset{<}{>} x_2$$

$$\Rightarrow f(x_1) \underset{<}{>} f(x_2)$$