MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 23

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Mean Value Thm

Suppose

f is continuous on $[a,b]$

f is differentiable on (a,b)

Then $\exists\ c \in (a,b)$ s.t.

$$f'(c) = \frac{f(a)-f(b)}{a-b} = \frac{f(b)-f(a)}{b-a}$$

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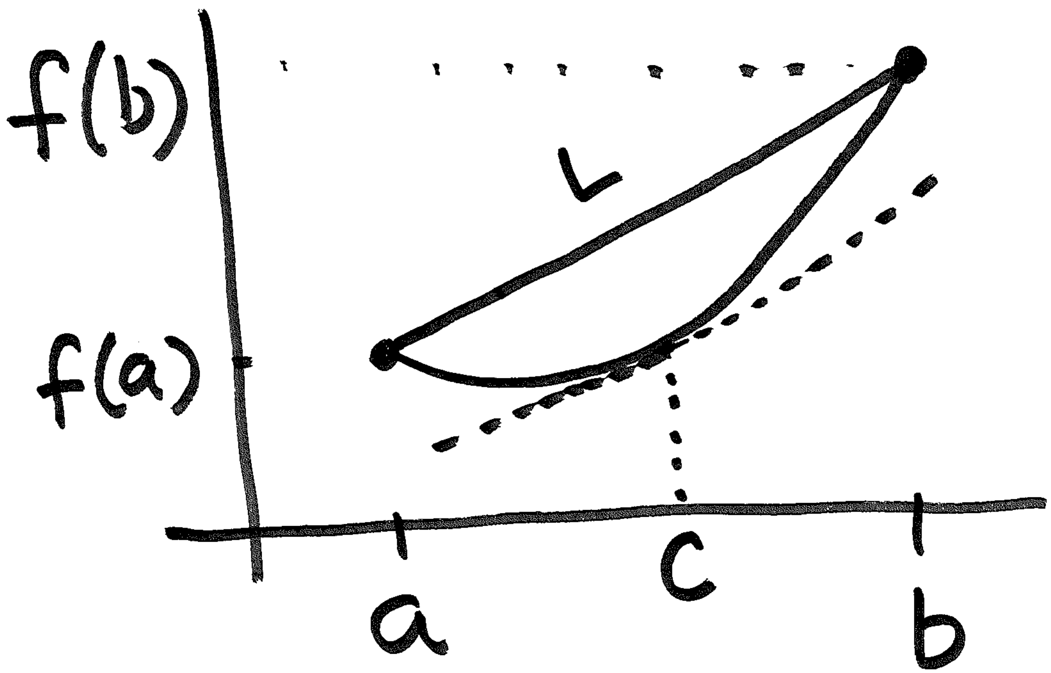

L : line joining $(a, f(a))$ & $(b, f(b))$

Slope of L $= \dfrac{f(a)-f(b)}{a-b}$

$f'(c)$ = slope of the tangent at c = slope of L

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Proof of the MVT

Let

$$g(x) = f(x) - \frac{f(b) - f(a)}{b-a} x$$

$$g(b) = f(b) - \frac{f(b) - f(a)}{b-a} b$$

$$= \frac{b\cancel{f(b)} - af(b) - b\cancel{f(b)} + bf(a)}{b-a}$$

$$= \frac{bf(a) - af(b)}{b-a}$$

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$$g(a) = f(a) - \frac{f(b)-f(a)}{b-a}\,a$$

$$= \frac{bf(a) - a\cancel{f(a)} - af(b) + \cancel{af(a)}}{b-a}$$

$$= \frac{bf(a) - af(b)}{b-a}$$

Thus

$$g(b) = g(a).$$

Further, g is continuous on $[a,b]$ and differentiable on (a,b).
By Rolle's Thm (last class) $\exists\ c \in (a,b)$ such that

$g'(c) = 0$.

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

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$$g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

$\square$

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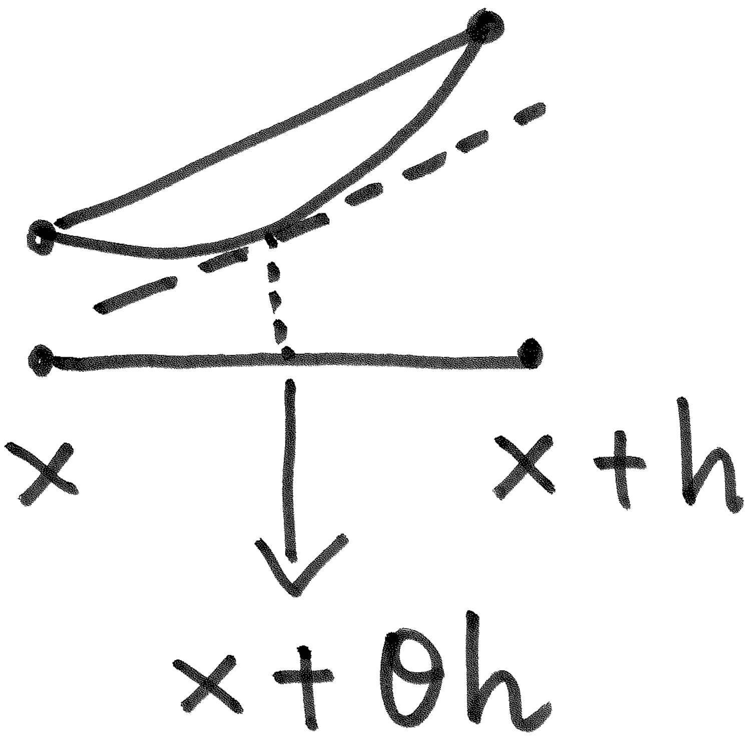
Another way to write the conclusion of the MVT:

$\exists\ \theta$ such that

$$f(x+h) - f(x) = h f'(x+\theta h)$$

$$0 < \theta < 1$$

$$\frac{f(\overset{b}{x+h}) - f(\overset{a}{x})}{\underset{b-a}{h}} = f'(\overset{c}{x+\theta h})$$



$0<\theta<1$

θ: fraction

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Corollary of the MVT

Let $f:(a,b) \longrightarrow \mathbb{R}$ be differentiable. Suppose that $f'(x) > 0$ for all $x \in (a,b)$. Then f is strictly increasing.

<u>Proof</u>: Let $u, v \in (a,b)$ s.t. $u < v$. Consider $[u,v]$. Then f is cont. on $[u,v]$, and

differentiable on (u,v). By the mean value theorem $\exists\, x_0 \in (u,v)$ s.t.

$$f'(x_0) = \frac{f(v)-f(u)}{v-u} > 0$$

$v-u > 0$ (from the choice of u,v)

Thus $f(v)-f(u)$ must be > 0

$\Longrightarrow f(v) > f(u)$

$\square$

Higher-Order derivatives

$f : D \longrightarrow \mathbb{R}$. ~~Suppose~~ If (Let f be differentiable on D)

$$\lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} \quad \text{exists}$$

then f is said to have a second derivative at x_0. The above limit is denoted by $f''(x_0)$.

Similarly, we can define the third and other higher order derivatives of f.

Notation :

1st : $\frac{df(x)}{dx}$, $f'(x)$, $f^{(1)}(x)$

2nd : $\frac{d^2f}{dx^2}$, $f''(x)$, $f^{(2)}(x)$.

$\vdots$

n th d^nf/dx^n, $f^{(n)}(x)$