MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 25

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Cauchy's Mean Value Thm
(generalization of the MVT)

If f and g are continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof: Define $h(x)$ by

$$h(x) = f(x)[g(b)-g(a)] - g(x)[f(b)-f(a)]$$

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$$h(a) = f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)]$$

$$= f(a)\,g(b) - g(a)\,f(b)$$

$$h(b) = f(b)[g(b) - g(a)] - g(b)[f(b) - f(a)]$$

$$= -f(b)\,g(a) + g(b)\,f(a)$$

$$= h(a)$$

$h(x)$ is also continuous on $[a,b]$

" " " differentiable on (a,b).

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By Rolle's thm., $\exists\ c \in (a,b)$

$$h'(c) = 0$$

$$0 = h'(c) = f'(c)[g(b)-g(a)] - g'(c)[f(b)-f(a)]$$

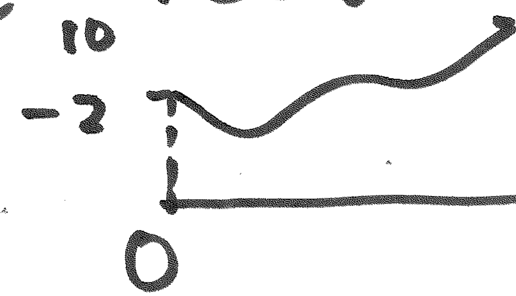
$$\Rightarrow f'(c)[g(b)-g(a)] = g'(c)[f(b)-f(a)]$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$\square$

Example

Show that $f(x) = 4x^5 + x^3 + 7x - 2$ has <u>exactly</u> one root.



$f(0) = -2$

$f(1) = 10$

$\underbrace{f(0)}_{-2} < 0 < \underbrace{f(1)}_{10}$

By the intermediate value thm $\exists\ c \in (0,1)$ s.t. $f(c) = 0$.

Thus, there is at least one root.

Suppose that there are 2 roots.

$\exists$ a, b s.t. $f(a) = f(b) = 0$.

f is continuous & differentiable on $[a, b]$. By Rolle's thm. $\exists$ c, $a < c < b$ s.t. $f'(c) = 0$

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$$f'(x) = 20x^4 + 3x^2 + 7$$

$$f'(c) = 20c^4 + 3c^2 + 7 \neq 0$$

for any c.

$\Rightarrow$ f has exactly one root.

Lipschitz functions.

A function is said to be Lipschitz if $\exists$ C s.t.

$$|f(x) - f(y)| \leq C|x-y|$$

$$\forall\ x, y \in D.$$

Ex: $f(x) = |x|$

f is not differentiable @ $x=0$.

f is Lipschitz.

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$f(x) = x$

f is differentiable &

f is also Lipschitz

$f'(x) = 1$

f' is bounded

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Theorem

Suppose f is differentiable on D. Then f' is bounded if and only if f is Lipschitz.

$\Longleftrightarrow$

Proof: ~~Lipsch~~ f' bounded $\Rightarrow f$ is Lipschitz

$|f'(x)| \leq M$ for some M. $\forall x \in D$.

Apply the MVT to (x, y):

$\exists\ c \in (x, y)$ s.t.

$$\frac{f(x) - f(y)}{x - y} = f'(c)$$

$$\Rightarrow |f(x) - f(y)| = |f'(c)||x-y|$$

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$\Rightarrow |f(x) - f(y)| \leq M|x-y|$

$\Rightarrow$ f is Lipschitz.

$\square$

Lipschitz $\Longrightarrow$ f' is bounded

(see HW #5)

f is differentiable w/ bounded derivative

$\Downarrow$ $\quad \Updownarrow$ if f is differentiable

Lipschitz $\Longrightarrow$ uniformly continuous

$\Longrightarrow$ continuity

$\not\Longrightarrow$ differentiability