MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 26

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Integration (Riemann)

$f : [a, b] \longrightarrow \mathbb{R}$

Is f integrable? (a bit later)

If f is integrable, then the integral of f, $\int_a^b f$, is a number assigned to f.

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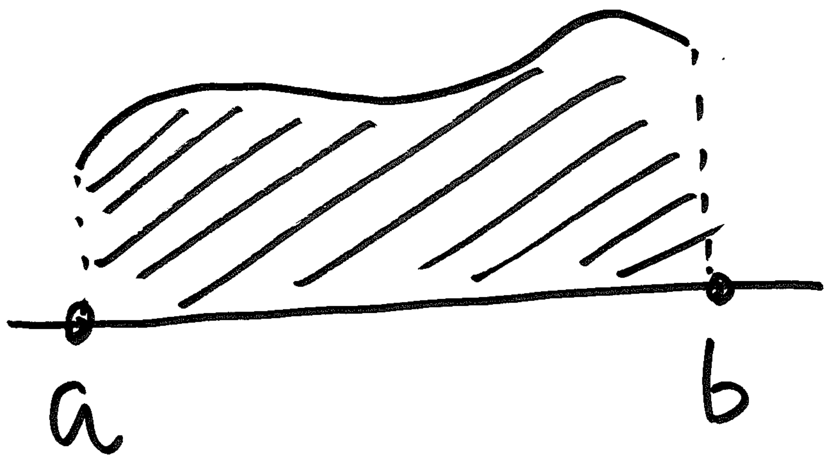
Motivation

$f \geqslant 0$



$\int_a^b f$ can be understood as the area of the region bounded above by the graph of ~~f~~ f & below by $[a, b]$.

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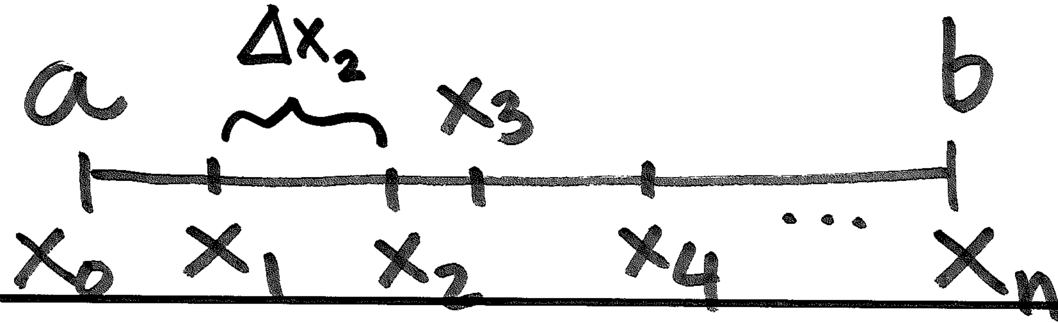
A <u>partition</u> P of a closed & bounded interval $[a, b]$ is a set of finitely many points $\{x_k\}_{k=0}^{n}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$



$\Delta x_2 = x_2 - x_1$

The length of the k th subinterval $[x_{k-1}, x_k]$ is

$$\Delta x_k = x_k - x_{k-1}.$$

Note:

$$\sum_{k=1}^{n} \Delta x_k = (\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + \cdots + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) + (x_n - \cancel{x_{n-1}})$$

$$= x_n - x_0 = b - a$$

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$f : [a,b] \longrightarrow \mathbb{R}$. Suppose that

f is bounded. $\exists\ m, M$ s.t.

$$m \leq f(x) \leq M \quad \forall x \in [a,b]$$

For each subinterval $[x_{k-1}, x_k]$ define:

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$$

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$$

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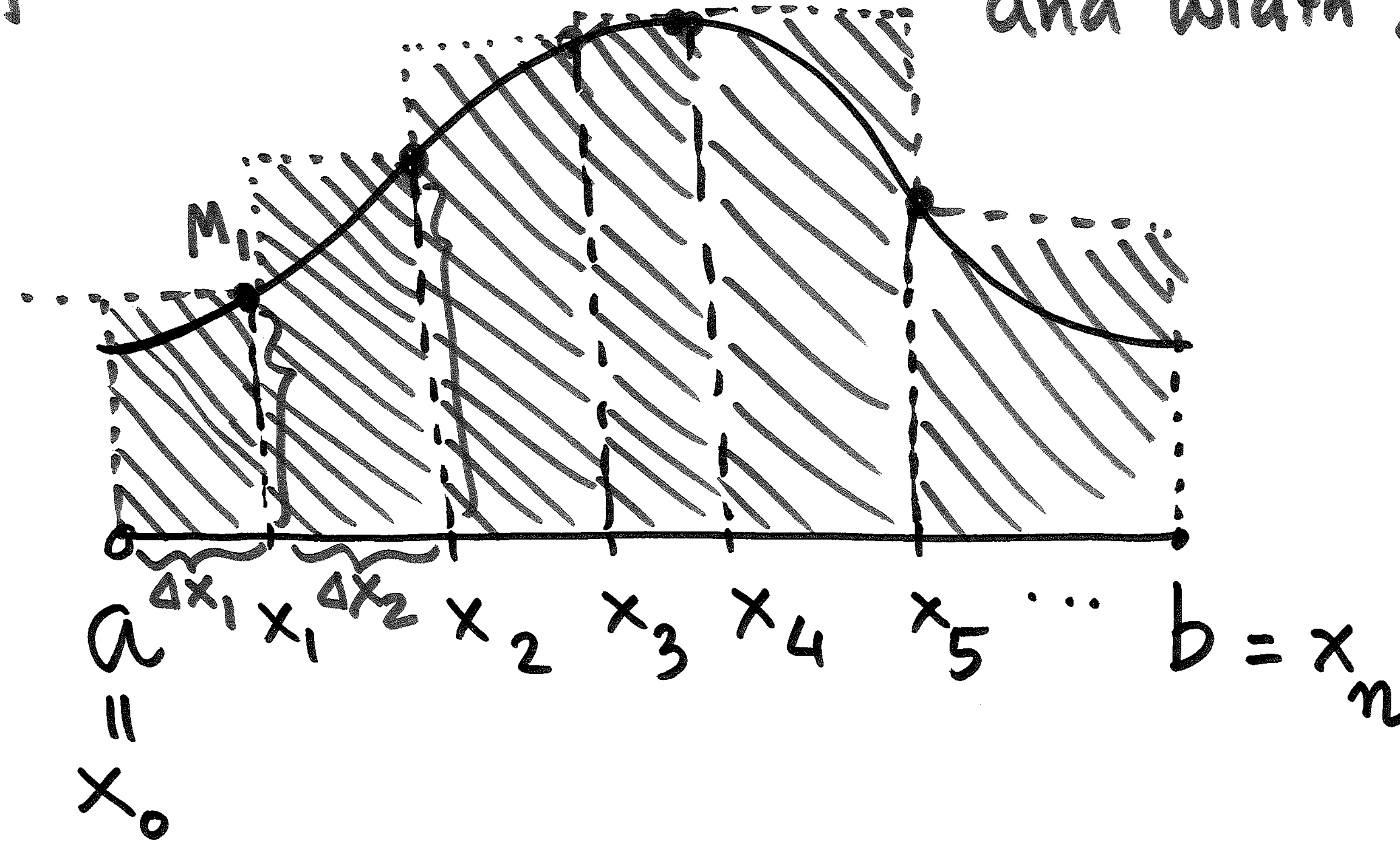
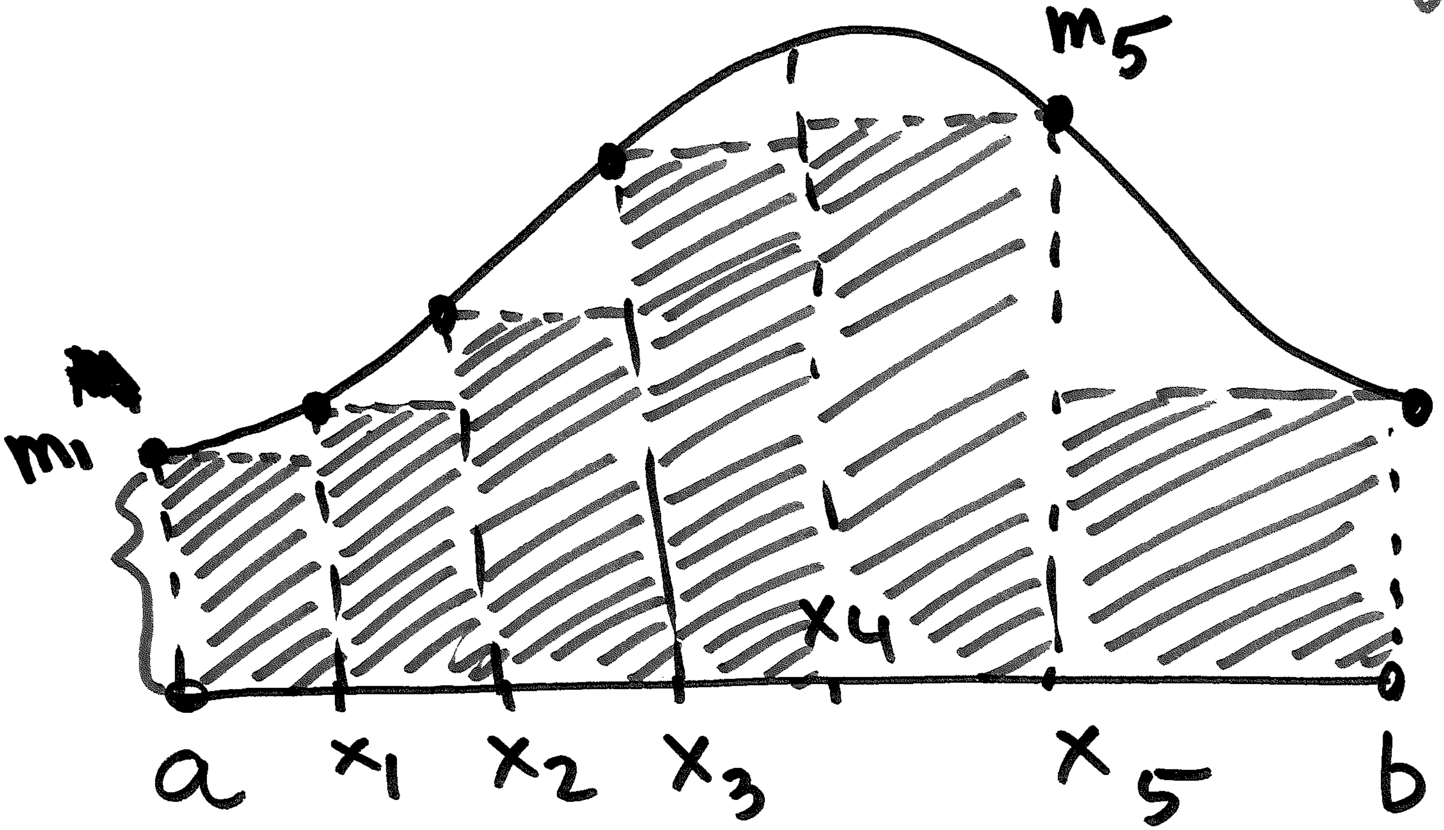
$U(P,f)$ = Sum of the areas of the rectangles with height M_k and width Δx_k.



$L(P,f)$ = sum of the areas of the rectangles with height m_k and width Δx_k.



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We define the upper sum as

$$U(P,f) = \sum_{k=1}^{n} M_k \Delta x_k$$

and the lower sum

$$L(P,f) = \sum_{k=1}^{n} m_k \Delta x_k$$

$U(P,f)$ & $L(P,f)$ depend on the partition P

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Example

$$f(x) = x^2 \ , \quad f : [0,2] \longrightarrow \mathbb{R}$$

Let $P = \left\{ 0, \frac{2}{n}, \frac{4}{n}, \dots, \underbrace{2 - \frac{2}{n}}_{= \frac{2(n-1)}{n}}, 2 \right\}$

$$\Delta x_k = \frac{2}{n} \ , \quad x_k = \frac{2k}{n} \qquad [\underset{m_k}{x_{k-1}}, \underset{M_k}{x_k}]$$

$$M_k = x_k^2 = \left(\frac{2k}{n}\right)^2 = 4k^2/n^2$$

$$m_k = x_{k-1}^2 = \left(\frac{2(k-1)}{n}\right)^2 = \frac{4}{n^2}(k-1)^2$$

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$$U(P,f) = \sum_{k=1}^{n} M_k \Delta x_k$$

$$= \sum_{k=1}^{n} 4 \frac{k^2}{n^2} \frac{2}{n} = \frac{8}{n^3} \sum_{k=1}^{n} k^2$$

$$= \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}$$

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$$L(P,f) = \sum_{k=1}^{n} m_k \Delta x_k$$

$$= \sum_{k=1}^{n} \frac{4}{n^2}(k-1)^2 \frac{2}{n}$$

$$= \frac{8}{n^3} \sum_{k=1}^{n} (k-1)^2$$

$$= \frac{8}{n^3} \sum_{j=1}^{n-1} j^2 = \frac{8}{n^3}(n-1)(n)(2(n-1)+1)$$

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$$= \frac{8}{n^3}(n-1)n\left(2(n-1)+1\right)$$

$$= \frac{4}{3n^3}(n-1)n(2n-1)$$

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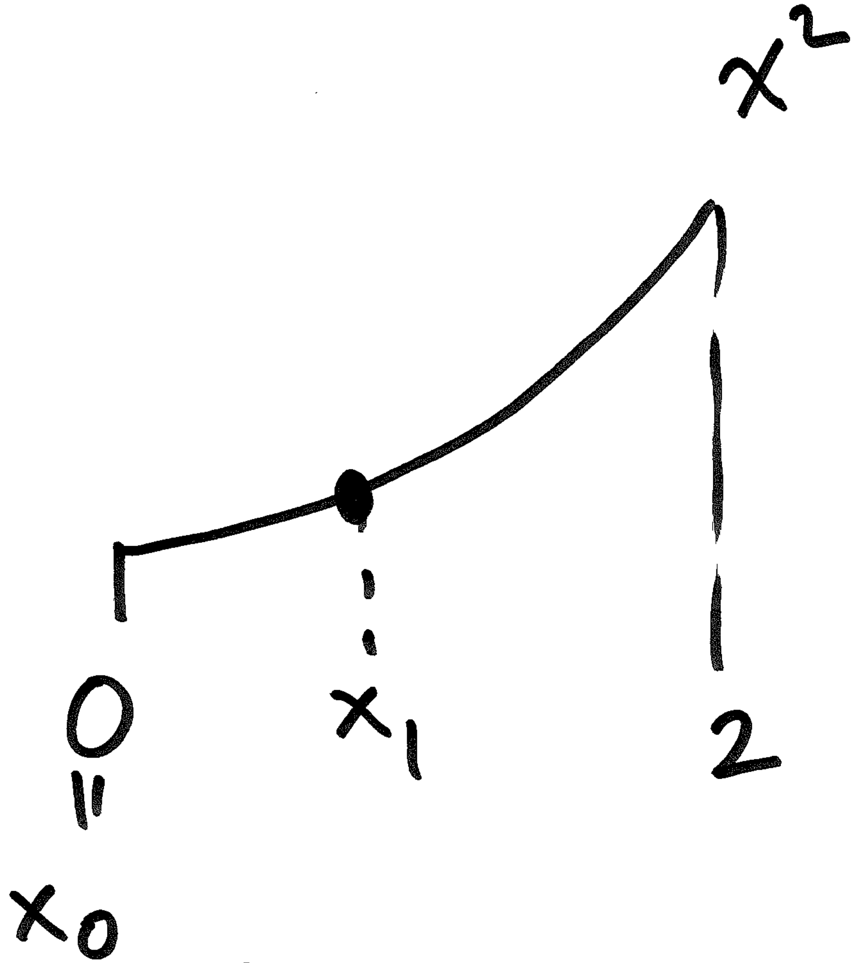

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$