MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 27

Darboux Lower & Upper sums

$f : [a, b] \longrightarrow \mathbb{R}$, f is bounded

$$P = \{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$$

$$x_0 < x_1 < x_2 < \cdots < x_n$$

$$m_i = \inf\{f(x) : x_{i-1} < x < x_i\}$$

$$M_i = \sup\{f(x) : x_{i-1} < x < x_i\}$$

$$\Delta x_i = x_i - x_{i-1}$$

Lower sum $L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$

Upper sum $U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$

<u>Lemma</u>: Let $f: [a,b] \to \mathbb{R}$ be bounded. $P = \{x_0, \dots, x_n\}$ is a partion of $[a,b]$. Then $\exists$ m, M s.t.

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

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Proof: For the kth subinterval

$$[x_{k-1}, x_k]:$$

$$m_k \le M_k$$

Since f is bounded, $\exists\ m, M$ s.t.

$$m \le f(x) \le M$$

inf

$m = \min f$ on $[a,b]$

$M = \max f$ on $[a,b]$

sup

4

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$$m \leq m_k \leq M_k \leq M$$

$$\Rightarrow m\,\Delta x_k \leq m_k \Delta x_k \leq M_k \Delta x_k \leq M \Delta x_k$$

$$\Rightarrow m \sum_{k=1}^{n} \Delta x_k \leq \sum_{k=1}^{n} m_k \Delta x_k \leq \sum_{k=1}^{n} M_k \Delta x_k$$

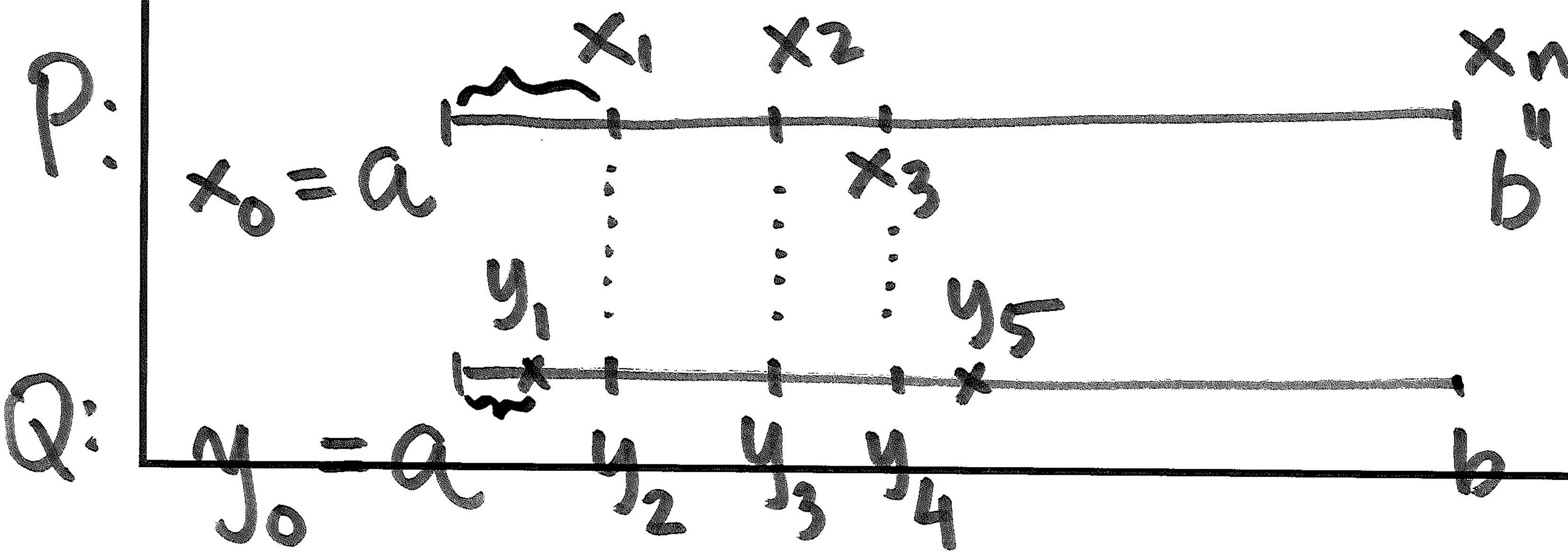
$$\leq M \sum_{k=1}^{n} \Delta x_k$$

$$\Rightarrow m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

$\square$

$P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$

Let Q be another partition of $[a,b]$ s.t. Q has all points of P and a finite no. of additional points.



$x_1 = y_2$

$x_2 = y_3$

$x_3 = y_4$

Q is called a <u>refinement</u> of P

Notation: $P \subseteq Q$

<u>Lemma</u>: (a) If $P \subseteq Q$, then

$$L(P,f) \leq L(Q,f)$$

&

$$U(P,f) \geq U(Q,f)$$

(b) $L(P,f) \leq U(Q,f)$

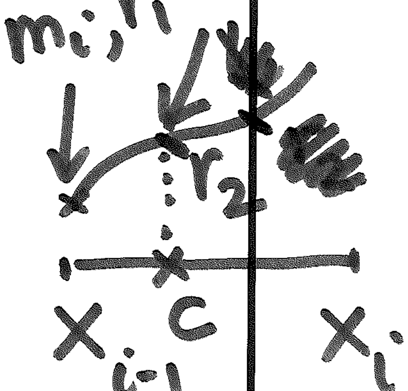
Proof:

$$P = \{ \underset{a}{\underset{\parallel}{x_0}}, x_1, x_2, \ldots, x_{i-1}, x_i, \ldots, \overset{b}{\overset{\parallel}{x_n}} \}$$

$$Q = \{ x_0, x_1, x_2, \ldots, x_{i-1}, c, x_i, \ldots, x_n \}$$

$P \subset Q$, Q has just one additional point.

On $[x_{i-1}, x_i]$



$m_i = \inf \{ f(x) : x_{i-1} \leq x \leq x_i \}$

$r_1 = \inf \{ f : x_{i-1} \leq x \leq c \}$

$r_2 = \inf \{ f : c \leq x \leq x_i \}$

$$m_i = \min(r_1, r_2)$$

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$$L(P,f) = \sum_{k=1}^{n} m_k \Delta x_k$$

$$= \sum_{k=1}^{i-1} m_k \Delta x_k + \underbrace{m_i (x_i - x_{i-1})}_{k=i} + \sum_{k=i+1}^{n} m_k \Delta x_k$$

$$= \sum_{k=1}^{i-1} m_k \Delta x_k + m_i (x_i - c + c - x_{i-1}) + \sum_{k=i+1}^{n} m_k \Delta x_k$$

$$\le \sum_{k=1}^{i-1} m_k \Delta x_k + r_2 (x_i - c) + r_1 (c - x_{i-1}) + \sum_{k=i+1}^{n} m_k \Delta x_k$$

$$= L(Q, f).$$

If there are n additional points in Q, then we continue the same argument n times.

(b) Already showed that

$$L(P,f) \underset{\text{part (a)}}{\leq} L(Q,f) \leq U(Q,f) \underset{\text{part (a)}}{\leq} U(P,f)$$

$$\Longrightarrow L(P,f) \leq U(Q,f).$$

$\square$