MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 28

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$f: [a,b] \longrightarrow \mathbb{R}$

As partitions of $[a,b]$ get "finer" the lower sums will increase whereas the upper sums will decrease. (see last lecture)

$$L(P,f) \leq U(P,f)$$

$$L(P,f) \leq L(Q,f) \leq U(Q,f) \leq U(P,f)$$

$P \subseteq Q$

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Lower & upper integrals

$$\underline{\int_a^b} f = \sup\{L(P,f) : P \text{ is any partition of } [a,b]\}$$

is defined to be the lower integral of f.

$$\overline{\int_a^b} f = \inf\{U(P,f) : P \text{ is any partition}\}$$

is the upper integral of f.

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$$L(P,f) \leq \sup_P L(P,f) \leq \inf_P U(P,f) \leq U(P,f)$$

$$\Rightarrow L(P,f) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(P,f)$$

(Riemann) Integrability

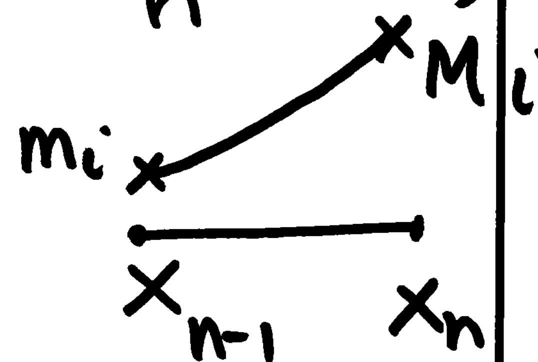
<u>Def</u>: A bounded function
$$f : [a,b] \longrightarrow \mathbb{R}$$
is (Riemann) integrable if
$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

The common value is denoted by $\int_a^b f$: the integral of f.

Example

$$f : [0, 2] \longrightarrow \mathbb{R} \qquad f(x) = x^2$$

$$P_n = \Big\{ 0, \underset{\substack{\parallel \\ x_1}}{\frac{2}{n}}, \underset{\substack{\parallel \\ x_2}}{\frac{4}{n}}, \cdots, \frac{2(n-1)}{n}, 2 \Big\}$$



$$L(P, f) = \frac{4}{3n^2}(n-1)(2n-1)$$

$$U(P, f) = \frac{4(n+1)(2n+1)}{3n^2}$$

$$\Delta x_k = x_k - x_{k-1} = \frac{2}{n}$$

The partition gets finer as $n \to \infty$.

$$\lim_{n\to\infty} L(P,f) = 8/3$$

$$\lim_{n\to\infty} U(P,f) = 8/3$$

$$\frac{8}{3} \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le \frac{8}{3}$$

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$$\Rightarrow \quad \underline{\int_a^b} f = \overline{\int_a^b} f = \frac{8}{3}$$

$\Rightarrow \quad f = x^2 : [0,2] \to \mathbb{R}$ is integrable

Verify $\int_0^2 x^2\,dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$.

Example

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$f : [-2, 3] \longrightarrow \mathbb{R}$ given by

Dirichlet function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 4 & \text{if } x \text{ is irrational} \end{cases}$$

$P = \{ \underset{\underset{-2}{\shortparallel}}{x_0}, x_1, x_2, \ldots, \underset{\underset{3}{\shortparallel}}{x_n} \}$ is a partition of $[-2, 3]$.

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For each subinterval $[x_{k-1}, x_k]$

$m_k = 0 \quad , M_k = 4$

$L(P,f) = 0$

$U(P,f) = \sum_{k=1}^{n} 4 \Delta x_k$

$= 4 \sum \Delta x_k = 4(3-(-2))$

$= 20$

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$$\underline{\int_a^b} f = 0 \quad , \quad \overline{\int_a^b} f = 20$$

$$\Rightarrow \underline{\int_a^b} f \neq \overline{\int_a^b} f$$

$\Rightarrow$ f is NOT integrable.

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Theorem

Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is (Riemann) integrable $\iff$ $\exists$ a sequence $\{P_n\}$ of $[a,b]$ such that

$$\lim_{n \to \infty} \left[U(f, P_n) - L(f, P_n) \right] = 0.$$

(Proof in the next lecture)

Scratch

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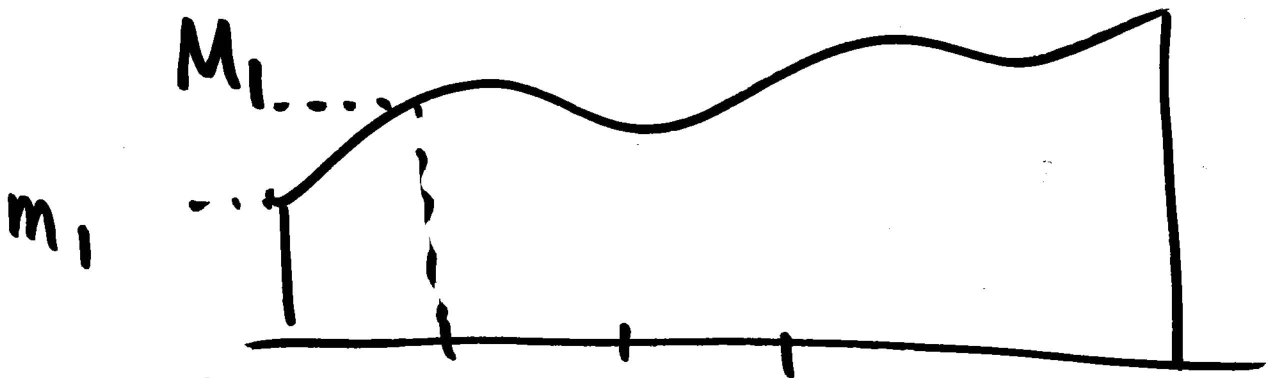

$$\{a \quad x_1 \quad x_2 \quad x_3 \quad \cdots \quad b\} = P$$

$$L = \sum m_i \Delta x_i$$

$$U = \sum M_i \Delta x_i$$

$$\underline{\int_a^b} f = \sup_P \{L(P,f)\}, \quad \overline{\int_a^b} f = \inf_P \{U(P,f)\}$$

$$[0, 2]$$

$$P_1 = \{0, 2\}$$

$$P_2 = \{0, 1, 2\}$$

$$P_3 = \{0, 2/3, 4/3, \ldots\}$$