MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 29

Archimedes Riemann Thm

Let $f : [a,b] \to \mathbb{R}$ be bounded.

f is integrable $\Longleftrightarrow$ $\exists$ a sequence $\{P_n\}_{n=1}^{\infty}$ of partitions $[a,b]$ such that

$$\lim_{n\to\infty} \left[U(P_n, f) - L(P_n, f) \right] = 0$$

2

Proof

University of Idaho

($\Rightarrow$) Assume that f is integrable.

Then $$\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$$

$$\underline{\int_a^b} f = \sup_P L(P,f)$$ sup = l.u.b.

For some $n \in \mathbb{N}$, $\underline{\int_a^b} f - \frac{1}{n}$

is no longer an upper bound.

Therefore, $\exists$ a partition P' s.t.

$$\underline{\int_a^b} f - \frac{1}{n} < L(P', f)$$

$$\Rightarrow \int_a^b f - \frac{1}{n} < L(P', f)$$

$$\overline{\int_a^b} f = \inf_P U(P, f) \qquad \inf = g.l.b$$

$\overline{\int_a^b} f + \frac{1}{n}$ is no longer a lower bound.

Then $\exists$ a partition P'' s.t.

$$U(P'', f) < \overline{\int_a^b} f + \frac{1}{n}$$

$$\Rightarrow U(P'', f) < \int_a^b f + \frac{1}{n}.$$

Let P_n be a common refinement of P' and P''.

$$P_n = P' \cup P''$$

University of Idaho

a previous result

Then

(1) $$\int_a^b f - \frac{1}{n} < L(P',f) < L(P_n,f)$$

and

(2) $$U(P_n,f) < U(P'',f) < \int_a^b f + \frac{1}{n}$$

Combining (1) & (2)

$$\int_a^b f - \frac{1}{n} < L(P_n, f) < U(P_n, f) < \int_a^b f + \frac{1}{n}$$

$$\Rightarrow 0 < U(P_n, f) - L(P_n, f) < \frac{2}{n}$$

Take lim as $n \to \infty$

$$0 < \lim_{n \to \infty} \left[U(P_n, f) - L(P_n, f) \right] < \lim_{n \to \infty} \frac{2}{n} = 0$$

$$\Rightarrow \lim_{n \to \infty} \left[U(P_n, f) - L(P_n, f) \right] = 0$$

8

University of Idaho

($\Leftarrow$):

need to show that f is integrable i.e. $\underline{\int_a^b} f = \overline{\int_a^b} f$.

We already know

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

University of Idaho

For a fixed n,

$$L(P_n, f) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(P_n, f)$$

$$\Rightarrow 0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(P_n, f) - L(P_n, f)$$

$$\Rightarrow 0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le \lim_{n\to\infty} \left[U(P_n, f) - L(P_n, f) \right]$$

$$\Rightarrow 0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le 0$$

$$\Rightarrow \overline{\int_a^b} f - \underline{\int_a^b} f = 0$$

$$\Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f$$

$\Rightarrow$ f is integrable $\square$

11

University of Idaho

Application of Arch. Riemann Thm

<u>Thm</u> A monotone function $f: [a,b] \longrightarrow \mathbb{R}$ is integrable.

<u>Proof</u>: Let f be monotonically increasing. Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$.

12

University of Idaho

$$m_i = \inf\{f : x \in [x_{i-1}, x_i]\} = f(x_{i-1})$$

$$M_i = \sup\{f : x \in [x_{i-1}, x_i]\} = f(x_i)$$

For $n \in \mathbb{N}$, let P_n be

$$\{\underset{\overset{\|}{a}}{x_0}, a + \frac{(b-a)}{n}, a + \frac{2(b-a)}{n}, \dots, a + \frac{(n-1)(b-a)}{n}, \underset{\overset{\|}{x_n}}{b}\}$$

is equispaced

each sub-int of length $\frac{b-a}{n}$

13

University of Idaho

$$U(P_n, f) - L(P_n, f)$$

$$= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} \left[f(x_i) - f(x_{i-1}) \right] \frac{(b-a)}{n}$$

$$= \frac{(b-a)}{n} \sum_{i=1}^{n} \left[f(x_i) - f(x_{i-1}) \right]$$

14

University of Idaho

$$= \frac{b-a}{n}\left[f(b) - f(a)\right]$$

Take the limit as $n \to \infty$

$$\lim_{n\to\infty}\left[U(P_n, f) - L(P_n, f)\right]$$

$$= 0$$

$\Rightarrow f$ is integrable (by ~~Archimedes~~ Riemann Thm)

For f decreasing the argument is similar. $\square$