

MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 29

University of Idaho - Archimedes Riemann Thm

Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded.

f is integrable $\Leftrightarrow \exists$ a sequence
 $\{P_n\}_{n=1}^{\infty}$ of partitions $[a, b]$ such

that

$$\lim_{n \rightarrow \infty} [U(P_n, f) - L(P_n, f)] = 0$$

(\Rightarrow) Assume that f is integrable.

Then

$$\int_a^b f = \bar{\int}_a^b f = \int_a^b f$$

$$\int_a^b f = \sup_P L(P, f) \quad \text{sup = l.u.b.}$$

For some $n \in \mathbb{N}$, $\int_a^b f - \frac{1}{n}$

is no longer an upper bound.

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Therefore, \exists a partition P' 's. t.

$$\int_a^b f - \frac{1}{n} < L(P', f)$$

$$\Rightarrow \int_a^b f - \frac{1}{n} < L(P', f)$$

$$\bar{\int}_a^b f = \inf_P U(P, f) \quad \inf = g.l.b$$

$\bar{\int}_a^b f + \frac{1}{n}$ is no longer a lower bound.

Then \exists a partition P'' s.t.

$$U(P'', f) < \int_a^b f + \frac{1}{n}$$

$$\Rightarrow U(P'', f) < \int_a^b f + \frac{1}{n}.$$

Let P_n be a common refinement of P' and P'' .

$$P_n = P' \cup P''$$

a previous
result

Then

$$(1) \quad \int_a^b f - \frac{1}{n} < L(P', f) \leq L(P_n, f)$$

and

$$(2) \quad U(P_n, f) \leq U(P'', f) \leq \int_a^b f + \frac{1}{n}$$

Combining (1) & (2)

$$\int_a^b f - \frac{1}{n} < L(P_n, f) < U(P_n, f) <$$

$$\int_a^b f + \frac{1}{n}$$

$$\Rightarrow 0 < U(P_n, f) - L(P_n, f)$$

$$< \frac{2}{n}$$

Take \lim as $n \rightarrow \infty$

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$$0 < \lim_{n \rightarrow \infty} [U(P_n, f) - L(P_n, f)]$$

$$< \lim_{n \rightarrow \infty} \frac{2}{n}$$

||
= 0

$$\Rightarrow \lim_{n \rightarrow \infty} [U(P_n, f) - L(P_n, f)] = 0$$

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(\Leftarrow):

need to show that f_- is integrable i.e. $\int_a^b f_- = \int_a^b f$.

We already know

$$\int_a^b f_- \leq \int_a^b f$$

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For a fixed n ,

$$L(P_n, f) \leq \int_a^b f \leq \int_a^{-b} f \leq U(P_n, f)$$

$$\Rightarrow 0 \leq \int_a^b f - \int_a^b f \leq U(P_n, f) - L(P_n, f)$$

$$\Rightarrow 0 \leq \int_a^b f - \int_a^b f \leq \lim_{n \rightarrow \infty} [U(P_n, f) - L(P_n, f)]$$

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$$\Rightarrow 0 \leq \bar{\int}_a^b f - \underline{\int}_a^b f \leq 0$$

$$\Rightarrow \bar{\int}_a^b f - \underline{\int}_a^b f = 0$$

$$\Rightarrow \bar{\int}_a^b f = \underline{\int}_a^b f$$

\Rightarrow f is integrable



Thm A monotone function
 $f: [a, b] \rightarrow \mathbb{R}$ is integrable.

Proof: Let f be monotonically increasing. Let

$P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$.

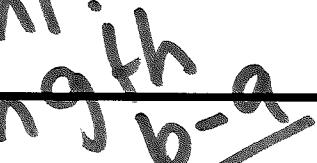
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$$m_i = \inf \{ f : x \in [x_{i-1}, x_i] \} = f(x_{i-1})$$

$$M_i = \sup \{ f : x \in [x_{i-1}, x_i] \} = f(x_i)$$

For $n \in \mathbb{N}$, let P_n be

$$\left\{ x_0, a + \frac{(b-a)}{n}, a + \frac{2(b-a)}{n}, \dots, \right.$$

a 
equispaced 
each sub-int. is 
of length 

$$\left. a + \frac{(n-1)(b-a)}{n}, b \right\}$$

$$\begin{aligned} & U(P_n, f) - L(P_n, f) \\ &= \sum_{i=1}^n (M_i - m_i) (x_i - x_{i-1}) \\ &= \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \frac{(b-a)}{n} \\ &= \frac{(b-a)}{n} \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \end{aligned}$$

$$\frac{b-a}{n} [f(b) - f(a)]$$

Take the limit as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} [V(P_n, f) - L(P_n, f)]$$

$$= 0$$

$\Rightarrow f$ is integrable (by Archimedes Riemann Thm)

For f decreasing the argument is similar. \square