MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 30

Recall from last lecture

Archimedes Riemann Thm:

$f : [a,b] \to \mathbb{R}$ is bounded.

f is integrable $\Longleftrightarrow$ $\exists$ $\underline{\underline{a}}$ sequence of partitions $\{P_n\}$ s.t.

$$\lim_{n \to \infty} \left[U(P_n, f) - L(P_n, f) \right] = 0 .$$

$$\lim_{n \to \infty} U(P_n, f) = \lim_{n \to \infty} L(P_n, f)$$

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Consequence of the Arc-Riem Thm

$$L(P_n,f) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(P_n,f)$$

If f is integrable:

$$L(P_n,f) \le \int_a^b f \le U(P_n,f)$$

Take the limit as $n \to \infty$

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$$\lim_{n\to\infty} L(P_n,f) \underset{=}{\le} \int_a^b f \underset{=}{\le} \lim_{n\to\infty} U(P_n,f)$$

same if f is integrable
due to the Archimedes-Riemann Thm

$$\Rightarrow \boxed{\int_a^b f = \lim_{n\to\infty} U(P_n,f)}$$

or,

$$\boxed{\int_a^b f = \lim_{n\to\infty} L(P_n,f)}$$

A sequence of partitions $\{P_n\}$ is said to be an <u>Archimedes</u> <u>sequence of partitions</u> of f provided

$$\lim_{n\to\infty}\left[U(P_n,f)-L(P_n,f)\right]=0$$

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Properties of integration

(I) If $f : [a, b] \to \mathbb{R}$ is integrable and C is some number in $\mathbb{R}$, then

$$\int_a^b cf = C \int_a^b f$$

(II) Linearity of the integral:
$f, g : [a, b] \to \mathbb{R}$ are integrable
Then for any $\alpha, \beta \in \mathbb{R}$

$$\int_a^b [\alpha f + \beta g] = \alpha \int_a^b f + \beta \int_a^b g$$

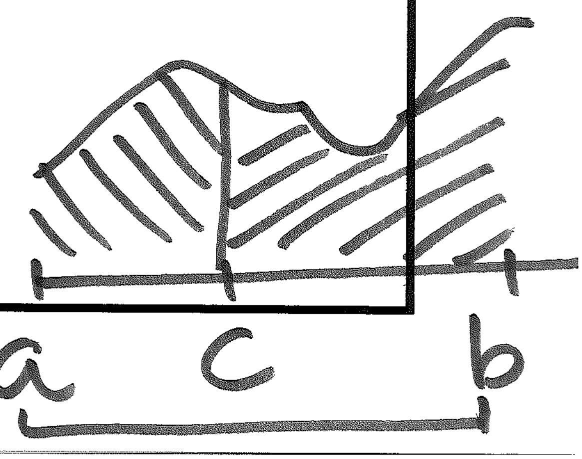
(III) Additivity over intervals :

$f : [a, b] \to \mathbb{R}$ is integrable

Let $c \in (a, b)$ $a < c < b$

Then

$$\int_a^b f = \int_a^c f + \int_c^b f$$



(IV) $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ are integrable, $f(x) \le g(x)$.

Then

$$\int_a^b f \le \int_a^b g$$

<u>Proof</u>: Let $\{P_n\}$ be an Archimedes sequence of

partitions for both f and g.

Since f & g are both integrable

due to the Archimedes Riemann thm

$$\lim_{n\to\infty} U(P_n, f) = \int_a^b f$$

$$\lim_{n\to\infty} U(P_n, g) = \int_a^b g$$

Since $f \leq g$

$$U(f, P_n) \leq U(P_n, g)$$

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$$\int_a^b f = \lim_{n\to\infty} U(f, P_n) \le \lim_{n\to\infty} U(P_n, g) = \int_a^b g$$

since f is integrable

Since g is integrable

$$\implies \int_a^b f \le \int_a^b g$$

$\square$

$f : [a,b] \to \mathbb{R}$ is integrable.

Then for any $c \in [a,b]$

$$\int_c^c f = 0$$

and if $c, d \in [a,b]$, $c < d$

then $$\int_c^d f = -\int_d^c f$$

Step functions

A function $f:[a,b]\to\mathbb{R}$ is called a step function if there is a partition

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

such that on each subinterval (x_{i-1}, x_i) the function f is a constant

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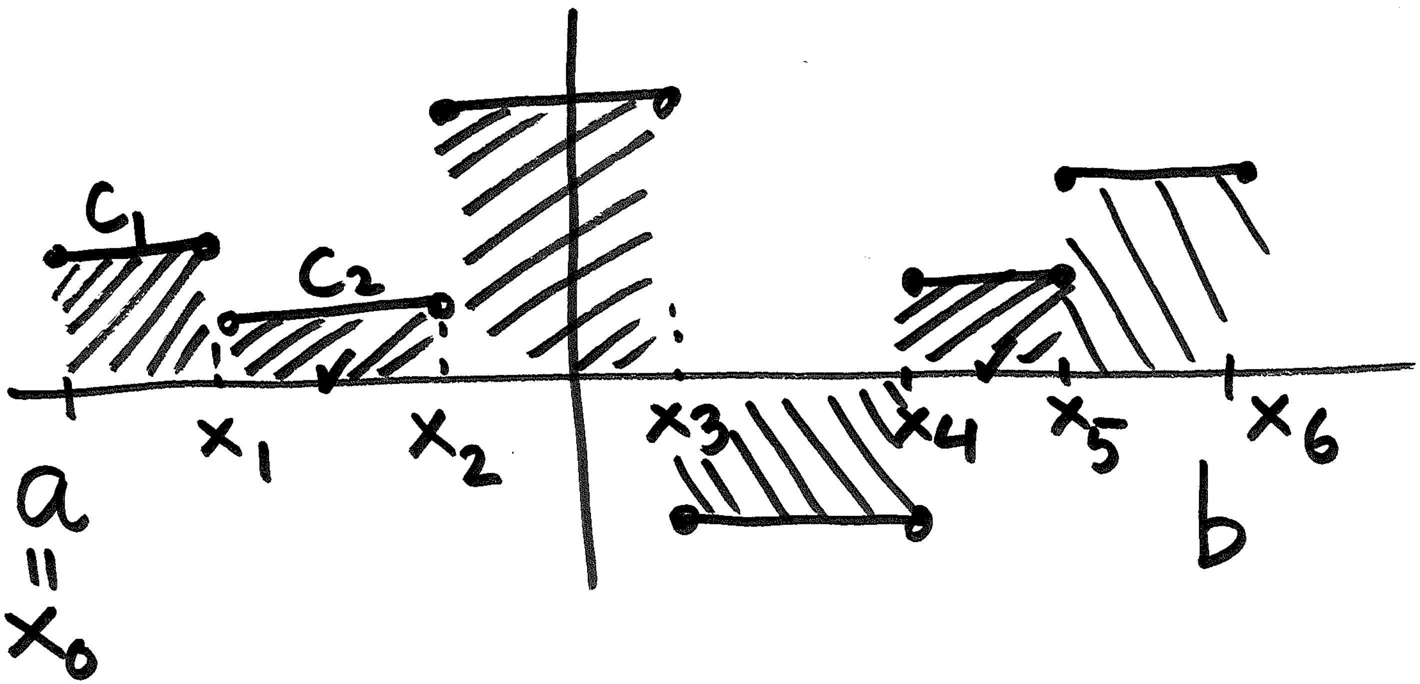
$$f(x) = C_i \quad \text{for } x \in (x_{i-1}, x_i)$$



(*) Step functions are integrable

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If f is a step-function then

$$\int_a^b f = \sum_{i=1}^{n} c_i (x_i - x_{i-1}).$$

Note: Boundedness ~~is~~ is needed for integrability but not continuity. Step functions ~~are~~ need not be continuous.

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Let $\{Q_n\}$ is an Arc. seq. for f and $\{R_n\}$ is an Arc. seq. for g.

Consider $P_n = Q_n \cup R_n$

P_n is a refinement and therefore an Arc. seq for both f & g