MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 31

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# Archimedes-Riemann Thm

$f : [a,b] \to \mathbb{R}$, $f$ is bounded

$f$ is integrable $\iff$ $\exists$ a seq of partitions $\{P_n\}_{n=1}^{\infty}$ of $[a,b]$ s.t. $\lim\limits_{n\to\infty} \left[U(P_n,f) - L(P_n,f)\right] = 0$.

or, given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t.

$$|U(P_n,f) - L(P_n,f)| < \varepsilon, \quad n \geq N$$

or, $\exists$ a partition $P$ s.t.

$$|U(P,f) - L(P,f)| < \varepsilon$$

Let $P = P_N$ or $P = P_{N+4}$

Reformulate Arc.-Riem. Thm:

$f$ is integrable $\Longleftrightarrow$ given $\varepsilon$ $\exists$ a partition $P$ s.t.

$$U(P,f) - L(P,f) < \varepsilon.$$

Recall:



$f$ is continuous at $x_0$ if whenever a sequence $\{x_n\}$ is such that $\{x_n\} \to x_0$ then $\{f(x_n)\} \to f(x_0)$.

another def of continuity

Same as saying: given $\varepsilon > 0$ $\exists\ \delta$ such that whenever $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$.

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# Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ is (Riemann) integrable.

<u>Proof</u>: Since $f$ is continuous on a closed & bounded interval $f$ is uniformly continuous.

Given $\varepsilon$, $\exists\ \delta$ such that for all $x, t \in [a, b]$

$$|f(x) - f(t)| < \varepsilon \text{ whenever } |x-t| < \delta.$$

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$.

$$|P| = \max_{1 \le k \le n} |x_k - x_{k-1}|$$

$|P|$ is the length of the largest subinterval in $P$

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Pick a partition $P$ s.t. $|P| < \delta$.
By the extreme value theorem $\exists$ $t_k$ and $s_k$ in $[x_{k-1}, x_k]$ s.t.
$M_k = f(t_k)$ and $m_k = f(s_k)$.
Since $|x_k - x_{k-1}| < \delta$ we have $|t_k - s_k| < \delta$ and so

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$0 \leq f(t_k) - f(s_k) < \varepsilon$, for $k = 1, 2, \ldots n$

$U(P,f) - L(P,f)$

$$= \sum_{k=1}^{n} \big[\underbrace{f(t_k) - f(s_k)}_{<\varepsilon}\big](x_k - x_{k-1})$$

$$< \varepsilon \sum_{k=1}^{n} (x_k - x_{k-1}) = \varepsilon(b-a)$$

$$= \varepsilon_1 \qquad \square$$

$\Rightarrow f$ is integrable by the Arc. Riemann thm.

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## Consequence of the prev. thm

If $f$ is bounded on $[a, b]$ and continuous on $(a, b)$ then $f$ is still integrable on $[a, b]$.

$\int_a^b f$ does not depend on the end points.

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$$f(x) = \begin{cases} \sin \frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

$|f(x)| < 1 \Rightarrow f$ is bounded

sin

$f$ is not cont at $x=0$

but $f$ is cont. on $(0,1)$

$\Rightarrow f$ is integrable on $[0,1]$.