MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 32

Mean Value Theorems

Thm (First Mean Value Theorem for Integrals)

Let $f : [a,b] \to \mathbb{R}$ be ~~is~~ continuous and $g : [a,b] \to \mathbb{R}$ is integrable and $g \geqslant 0$. Then $\exists\, c \in (a,b)$ such that

$$\int_a^b f g = f(c) \int_a^b g$$

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Proof

Since f is continuous by the Extreme Value Thm $\exists$ m & M s.t.

$$\underbrace{m}_{\text{min}} \le f(x) \le \underbrace{M}_{\text{max}}, \quad \forall x \in [a,b].$$

Since $g \geq 0$,

$$m\, g(x) \le f(x)\, g(x) \le M\, g(x)$$

Integrating throughout

$$m\int_a^b g \le \int_a^b fg \le M\int_a^b g$$

If $\int_a^b g = 0$, then $\int_a^b fg = 0$.
(due to the inequality above).
In this case the result holds.

Let $\int_a^b g \neq 0$.

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Then

$$m \le \frac{\int_a^b fg}{\int_a^b g} \le M$$

min(f) max(f)

some no.

Since f is Continuous, by the Intermediate Value Thm $\exists\, c \in (a,b)$ s.t.

$$f(c) = \frac{\int_a^b fg}{\int_a^b g}$$

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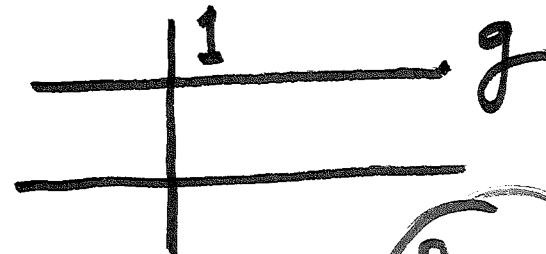
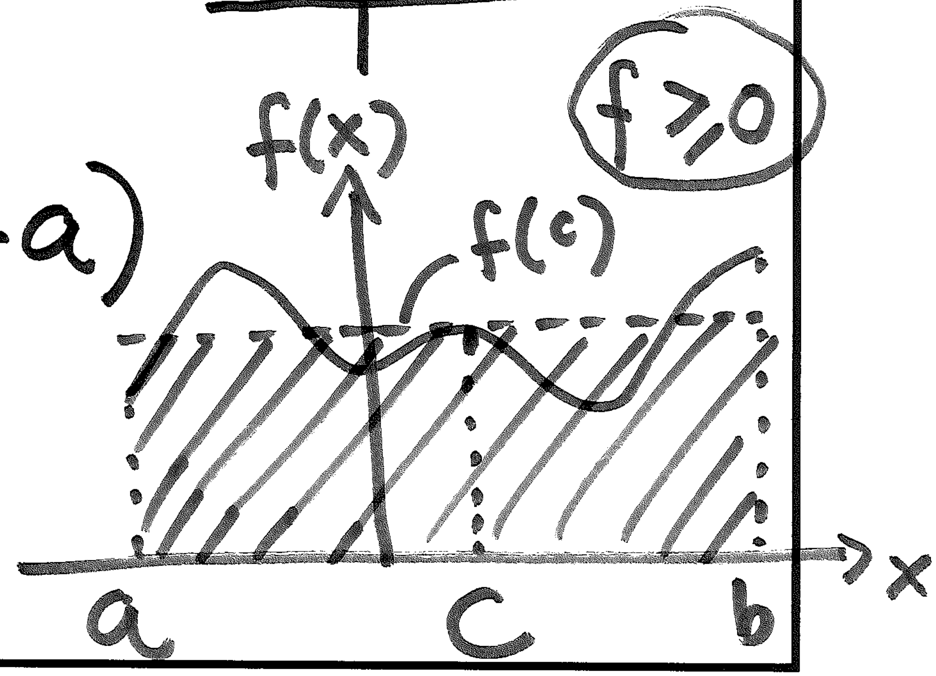
$$\Rightarrow \int_a^b fg = f(c)\int_a^b g \qquad \square.$$

COR: Let $g \equiv 1$.



$$\int_a^b f = f(c)\,(b-a)$$

area of the shaded region



Thm (Second Mean Value Thm of Integrals)

If a function $f : [a,b] \to \mathbb{R}$ is monotone, then $\exists\, c \in (a,b)$ such that

$$\int_a^b f = f(a)(c-a) + f(b)(b-c)$$

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Proof

Since f is monotone, it is integrable. Define $g(x)$ as

$$g(x) = f(a)(x-a) + f(b)(b-x)$$

g is a polynomial of degree 1 and hence <u>continuous</u>.

$$g(a) = f(b)(b-a); \quad g(b) = f(a)(b-a)$$

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$f(a) \leq f(x) \leq f(b)$

if f is monotonically increasing.

Integrate to get

$$f(a)(b-a) \leq \int_a^b f \leq f(b)(b-a)$$

$$g(b) \leq \int_a^b f \leq g(a)$$

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If f is decreasing, then

$$g(a) \leq \int_a^b f \leq g(b)$$

Applying the Int. Value Thm to g we can say that $\exists$ c s.t. $c \in (a, b)$

$$g(c) = \int_a^b f$$

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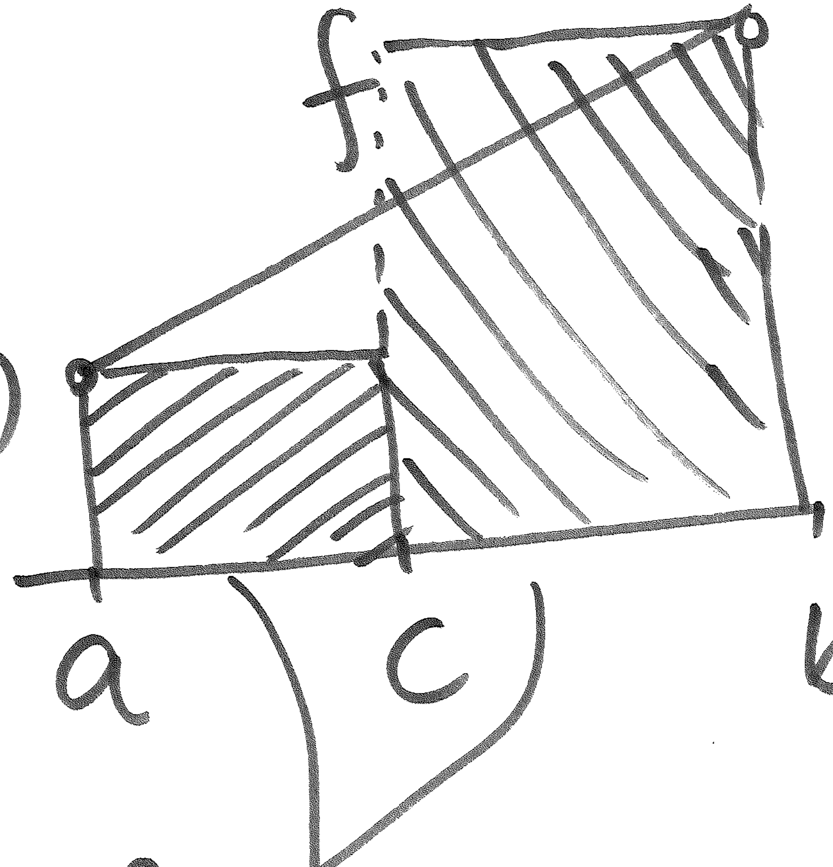
$$f(a)(c-a) + f(b)(b-c) = \int_a^b f$$

$\square$

$f \geq 0$



Sum of the areas of the 2 shaded regions equals $\int f$