MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 33

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Fundamental Theorems of Calculus

Given f.

Ques 1: Does there exist F such that $F'(x) = f(x)$?

$F \longrightarrow$ (is differentiable)

Ques 2. Under what conditions would f be the derivative of some F?

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If F exists, then it is called the anti-derivative of f.

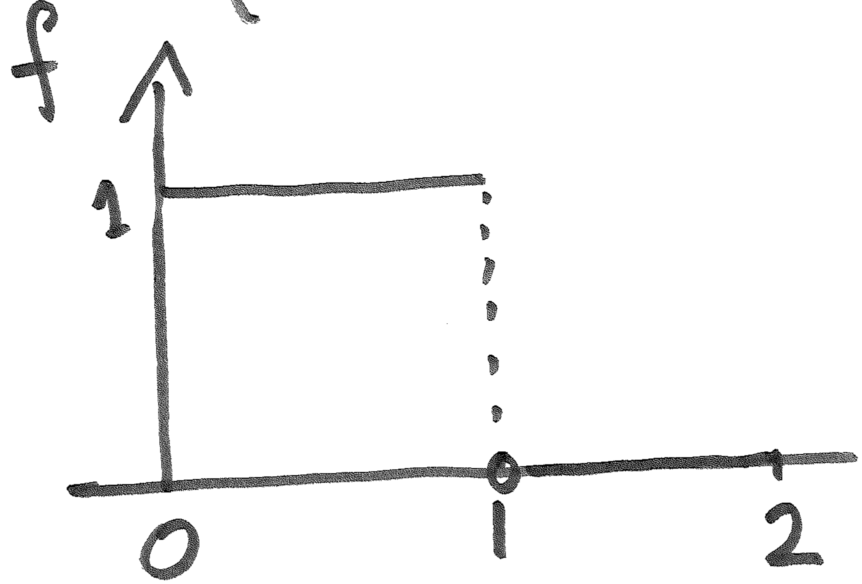
Not all fs have an anti-derivative.

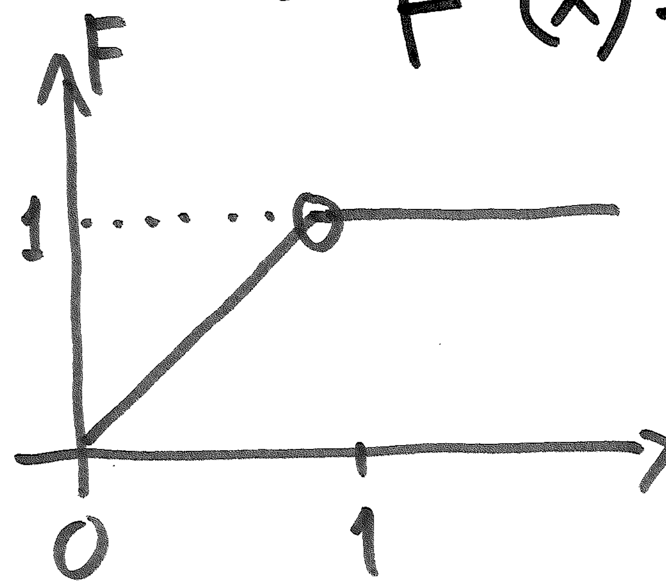
Step functions :

$$f = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & 1 < x \le 2 \end{cases}$$

not continuous

There does not exist F s.t. $F'(x) = f(x)$





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Let $F(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & 1 < x \le 2 \end{cases}$

But F is not differentiable at $x = 1$. Thus F cannot be an antiderivative of f.

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Antiderivative :

Given f , another function F is the anti-derivative of f if

$$F'(x) = f(x)$$

for all x

Fundamental Thm of Calc

If f is continuous on a closed & bounded interval $[a,b]$ and if

$$F(x) = \int_a^x f(t)\,dt$$

then

$$F'(x) = f(x).$$

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Proof

Take a fixed $x \in [a,b]$ and $h \neq 0$ s.t. $x+h \in [a,b]$.

$$F(x+h) - F(x)$$

$$= \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt$$

Assume that $h > 0$ (make minor adjustments for $h < 0$).

Since f is continuous on $[x, x+h]$, f attains a max M and a min ~~max~~ m in $[x, x+h]$.

$$m \leq f \leq M$$

Integrate from x to $x+h$:

$$mh \le \int_x^{x+h} f \le Mh$$

$$\Rightarrow m \le \frac{\int_x^{x+h} f}{h} \le M$$

Since f is continuous, by

the Intermediate Value Thm

$\exists\ c \in [x, x+h]$ s.t.

$$f(c) = \frac{\int_x^{x+h} f}{h} = \frac{F(x+h)-F(x)}{h}$$

Take the limit as $h \to 0$

$$\lim_{h\to 0} f(c) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$$

$$\Rightarrow f(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$$

this now exists

$$\Rightarrow f(x) = F'(x)$$

$\square$

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Application:

Solve the DE

continuous

Unknown is $F(x)$.

Solve for $F(x)$ $\to$ $$\frac{dF}{dx} = f(x)$$

$$F(x_0) = y_0 \leftarrow$$ initial condition

If f is continuous then

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the solution of the DE is

$$\underbrace{F(x) = \int_{x_0}^{x} f(x)}_{\text{Fundamental Thm}} + F(x_0)$$

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$$F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0}$$

$$x - x_0 = h$$

$$\lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h}$$