

MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 34

University of Idaho

Practice Problems

#6 For x rational : $f(x) = 0$

Prove that

$$\int_a^b f \leq 0 \leq \int_a^b -f$$

Let $P = [x_0, x_1, \dots, x_{n-1}, x_n]$

For each subinterval $[x_{i-1}, x_i]$

$$m_i = \inf \{ f(x) : x_{i-1} \leq x \leq x_i \}$$

Each $[x_{i-1}, x_i]$ has a rational r_i s.t. $f(r_i) = 0$. Therefore,

$$m_i \leq 0 \quad \text{for } i=1, \dots, n$$

$$L(P, f) = \sum_{i=1}^n m_i \underbrace{\Delta x_i}_{\text{positive}} \leq 0.$$

3

University of Idaho

Sup = l.u.b

$$\int_a^b f = \sup_P L(P, f) \leq 0$$

$$M_i = \sup \{ f(x) : x_{i-1} \leq x \leq x_i \}$$

Since each $[x_{i-1}, x_i]$ contains
a rational

$$M_i > 0$$

4

University of Idaho

positive

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i$$

$$\geq 0$$

$$\int_a^b f = \inf_P U(P, f) \geq 0$$

5

University of Idaho

$$f: [0, 1] \rightarrow \mathbb{R}$$

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show,

$$\int_0^1 f = 0 \quad \& \quad \int_0^1 f \geq \frac{1}{2}$$

Let $P = [x_0, x_1, \dots, x_n]$

$$m_i = \inf f \text{ on } [x_{i-1}, x_i]$$

$= 0$ (since $[x_{i-1}, x_i]$
must have an irrational)

Therefore

$$L(P, f) = \sum 0 \cdot \Delta x_i = 0$$

$$\int_a^b f = \sup_P L(P, f) = 0$$

Consider

$$P_n = \left[x_0, \underbrace{\frac{1}{n}, \frac{2}{n}, \dots}_{0} \frac{i}{n}, \dots, \frac{n-1}{n}, 1 \right]$$

$$\Delta x_i = \frac{1}{n}, \quad x_i = \frac{i}{n}$$

$$M_i = \sup \left\{ f(x) : x \in \left[x_{i-1}, x_i \right] \right\}$$

$\geq ? \frac{i-1}{n}$

$$U(P_n, f) = \sum_{i=1}^n M_i \Delta x_i$$

$$\Leftarrow \sum_{i=1}^n \left(\frac{i-1}{n} \right) & \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{i=1}^n (i-1)$$

sum of the
first $n-1$
natural nos.

$$= \frac{1}{n^k} \frac{\alpha(n-1)}{2}$$

$$= \frac{1}{2} \frac{n-1}{n} \rightarrow \frac{1}{2}$$

$$\int_a^b f = \inf_P U(f, P) \geq \frac{1}{2}$$

$$P = x_0 < x_1 \cdots < x_n$$

$$\Delta x_i = x_i - x_{i-1}$$

$$L(P, f) = \sum_{i=1}^n m_i \Delta x_i \checkmark$$

$$= \sum_{i=1}^n m_i \sum_{i=1}^n \Delta x_i \times$$

$$= \sum_{i=1}^n m_i (b - a)$$

University of Idaho —

$$\begin{aligned} & \textcircled{(1)(2)} + \textcircled{(2)(3)} + \textcircled{(3)(4)} \\ = & 2 + 6 + 12 = 20 \end{aligned}$$

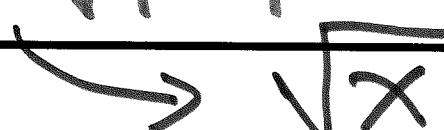
$$\begin{aligned} & (1+2+3) (2+3+4) \\ & \neq 20 \end{aligned}$$

3

$$g(x) = \sqrt{|x|} \quad \text{on } [-1, 1].$$

g is not differentiable at $x = 0$, so it does not satisfy the conditions of the MVT.

What if $g(x) = \sqrt{|x|}$ on $[\frac{1}{2}, 1]$?



1.

$$f(x) = \begin{cases} m_1 x + 4 & x \leq 0 \\ m_2 x + 4 & x \geq 0 \end{cases}$$

$m_1 \neq m_2$ f is not
differentiable at 0.

$$\lim_{\substack{x \rightarrow 0}} \frac{f(x) - f(0)}{x - 0} \quad ? \text{ exists?}$$

$x < 0$

$$\frac{f(x) - f(0)}{x - 0} = \frac{m_1 x + 4 - 4}{x}$$
$$= \frac{m_1 x}{x} = m_1$$

 $x > 0$:

$$\frac{f(x) - f(0)}{x - 0} = \frac{\cancel{m_1} x + \cancel{4} - 4}{\cancel{x}}$$
$$= m_2$$

$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$ does not

exist because it is
 m_1 if $x < 0$ and m_2 if
 $x > 0$ and $m_1 \neq m_2$.

$\Rightarrow f$ is not differentiable
at $x = 0$.