MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 34

Practice Problems

#6 For x rational: $f(x) = 0$

Prove that

$$\underline{\int_a^b} f \le 0 \le \overline{\int_a^b} f$$

Let $P = [\underset{\substack{\parallel \\ a}}{x_0}, x_1, \ldots, x_{n-1}, \underset{\substack{\parallel \\ b}}{x_n}]$

For each subinterval $[x_{i-1}, x_i]$

$$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$$

Each $[x_{i-1}, x_i]$ has a rational r_i s.t. $f(r_i) = 0$. Therefore,

$$m_i \leq 0 \quad \text{for } i = 1, \dots, n$$

$$L(P,f) = \sum_{i=1}^{n} m_i \underbrace{\Delta x_i}_{\text{positive}} \leq 0.$$

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$\sup = \text{l.u.b}$

$$\underline{\int_a^b} f = \sup_P L(P, f) \leq 0$$

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$$

Since each $[x_{i-1}, x_i]$ contains a rational

$$M_i \geqslant 0$$

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$$U(P,f) = \sum_{i=1}^{n} M_i \overbrace{\Delta x_i}^{\text{positive}}$$

$$\geq 0$$

$$\overline{\int_a^b} f = \inf_P U(P,f) \geq 0$$

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$f : [0,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show

$$\underline{\int_0^1} f = 0 \quad \& \quad \overline{\int_0^1} f \geqslant \frac{1}{2}$$

Let $P = [x_0, x_1, \dots, x_n]$

$m_i = \inf f$ on $[x_{i-1}, x_i]$

$= 0$ (since $[x_{i-1}, x_i]$ must have an irrational)

Therefore

$$L(P,f) = \sum 0 \cdot \Delta x_i = 0$$

$$\underline{\int_a^b} f = \sup_P L(P,f) = 0$$

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Consider

$$P_n = \left[\underset{\substack{\| \\ 0}}{x_0}, \underbrace{\tfrac{1}{n}, \tfrac{2}{n}}_{}, \ldots \tfrac{i}{n}, \ldots \tfrac{n-1}{n}, 1\right]$$

$1/n$

$$\Delta x_i = 1/n \quad , \quad x_i = i/n$$

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

$\frac{i-1}{n}$

$$\geq ? \ \frac{i-1}{n}$$

$$U(P_n, f) = \sum_{i=1}^{n} M_i \, \Delta x_i$$

$$\geq \sum_{i=1}^{n} \left(\frac{i-1}{n}\right) \cdot \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} (i-1)$$

sum of the first $n-1$ natural nos.

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$$= \frac{1}{n^{\cancel{2}}} \frac{\cancel{n}(n-1)}{2}$$

$$= \frac{1}{2} \frac{n-1}{n} \longrightarrow \frac{1}{2}$$

$$\overline{\int_a^b} f = \inf_P U(f, P) \geqslant \frac{1}{2}$$

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$$P = x_0 < x_1 \cdots < x_n$$

$$\Delta x_i = x_i - x_{i-1}$$

$$L(P,f) = \sum_{i=1}^{n} m_i \, \Delta x_i \quad \checkmark$$

$$= \sum_{i=1}^{n} m_i \sum_{i=1}^{n} \Delta x_i \quad \times$$

$$= \sum_{i=1}^{n} m_i \quad (b-a)$$

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$$(1)(2) + (2)(3) + (3)(4)$$

$$= 2 + 6 + 12 = 20$$

$$(1+2+3)(2+3+4)$$

$$\neq 20$$

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3 $g(x) = \sqrt{|x|}$ on $[-1, 1]$.

g is not differentiable at $x = 0$, so it does not satisfy the conditions of the MVT.

What if $g(x) = \sqrt{|x|}$ on $[\frac{1}{2}, 1]$?

$\rightarrow \sqrt{x}$

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1. $$f(x) = \begin{cases} m_1 x + 4 & x \leq 0 \\ m_2 x + 4 & x \geqslant 0 \end{cases}$$

$m_1 \neq m_2$ f is not differentiable at 0.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \text{ ? exists ?}$$

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$x < 0$

$$\frac{f(x)-f(0)}{x-0} = \frac{m_1 x + 4 - 4}{x}$$

$$= \frac{m_1 x}{x} = m_1$$

$x > 0$:

$$\frac{f(x)-f(0)}{x-0} = \frac{m_2 x + 4 - 4}{x}$$

$$= m_2$$

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$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist because it is m_1 if $x < 0$ and m_2 if $x > 0$ and $m_1 \neq m_2$.

$\Rightarrow$ f is not differentiable at $x = 0$.