MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 35

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Fundamental Theorems of Calculus

Theorem [Second Fundamental Thm]

$f : [a,b] \longrightarrow \mathbb{R}$, continuous.

If

$$F(x) = \int_a^x f(x)\,dx$$

then

$$F'(x) = f(x),$$

$$\forall\ x \in [a,b].$$

[See proof in a previous lecture]

Remark: 1) The FTOC guarantees an antiderivative F for every continuous f.

2) The antiderivatives of f differ from each other by a constant.

If $H(x) = F(x) + C$, then $H(x)$ is also an antiderivative of f.

$$H'(x) = F'(x)$$
$$= f(x)$$

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Application of the FTOC

Given f continuous. Consider solving

$$\left.\begin{aligned} \frac{dF}{dx}\ F'(x) &= f(x) \\ \text{subject to}\quad F(x_0) &= y_0 \end{aligned}\right\} \text{Solve for } F$$

initial condition

By the FTOC, if $F(x) = \int_{x_0}^{x} f(x)\,dx$

then $F'(x) = f(x)$

Set $F(x) = y_0 + \int_{x_0}^{x} f(x)\,dx$

Then $F'(x) = f(x)$

and $F(x_0) = y_0 + \int_{x_0}^{x_0} f(x)\,dx$

$= y_0$

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The solution is uniquely given by

$$F(x) = y_0 + \int_{x_0}^{x} f(x)\,dx$$

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Theorem [First Fundamental Thm]

If f is continuous $(f : [a,b] \to \mathbb{R})$

and if

$$\phi'(x) = f(x) \quad (a \leq x \leq b)$$

then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

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Recall (from past HWs)

Result 1: If $f'(x) = 0$ for $x \in [a, b]$ then f is constant on $[a, b]$.

Result 2: If $f'(x) = g'(x)$ for $x \in [a, b]$ then

$$f - g = \text{constant}$$

or $$f(x) = g(x) + C$$

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Proof

Let $F(x) = \int_a^x f(x)\,dx$

Then by the previous thm

$$F'(x) = f(x)$$

Since $\varphi'(x) = f(x)$ (is given)

we can say

$$F(x) = \varphi(x) + C$$

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$$F(b) - F(a) = \varphi(b) + \cancel{C} - [\varphi(a) + \cancel{C}]$$

$$= \varphi(b) - \varphi(a)$$

$$F(b) = \int_a^b f(x)\,dx \;;\; F(a) = 0$$

$$\Rightarrow \int_a^b f(x)\,dx = \varphi(b) - \varphi(a)$$

$\square$

2nd Application: Integration by parts

<u>Thm</u>: Suppose $f, g : [a,b] \to \mathbb{R}$ are continuously differentiable (f, g are differentiable and f', g' are continuous)

Then

$$\int_a^b (fg')(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b (f'g)(x)\,dx$$

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Let $h = fg$. h is differentiable

By the ~~chain~~ product rule

$$h' = f'g + fg'$$

h' is continuous (due to the assumptions) and hence integrable.

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$$\int_a^b h'(x)\,dx = \int_a^b f'g + \int_a^b f g'$$

By the fundamental thm

$$\int_a^b h'(x)\,dx = h(b) - h(a)$$

$$= f(b)g(b) - f(a)g(a)$$

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$$\int_a^b f'g + \int_a^b fg' = f(b)g(b) - f(a)g(a)$$

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g$$

$\square$

Example

Integrate $\displaystyle\int_1^2 \underbrace{\ln x}_{f} \cdot \underbrace{1}_{g'}\, dx$

$$g' = 1 \Rightarrow g = x$$

$$f(x) = \ln x \Rightarrow f' = \frac{1}{x}$$

$$\int_1^2 \ln x = x \ln x \Big|_{x=2} - x \ln x \Big|_{x=1} - \int_1^2 \frac{1}{x} x \, dx$$

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$$\int_1^2 \ln x = 2\ln 2 - 1\ln 1 - \int_1^2 dx$$

$$= 2\ln 2 - x\Big|_1^2$$

$$= 2\ln 2 - 1$$

$\square$