MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 36

Recall: 2nd FTOC

$f : [a,b] \to \mathbb{R}$, continuous

If $F(x) = \int_a^x f(t)\,dt$

then $F'(x) = f(x)$

Note: F is continuous & differentiable

Rewrite: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

University of Idaho

Corollary of the FTOC

$f : [a, b] \to \mathbb{R}$ is continuous.

Take a fixed point $x_0 \in [a, b]$.

Then

$$\frac{d}{dx} \int_{x_0}^{x} f = f(x), \quad \forall x \in [a, b]$$

Proof:

$$\frac{d}{dx}\left[\int_{x_0}^{x} f\right] = \frac{d}{dx}\left[\int_{x_0}^{a} f + \int_{a}^{x} f\right]$$

$$= \frac{d}{dx}\left[\int_{a}^{x} f\right]$$

$$= f(x)$$

$\square$

University of Idaho

Another Corollary

$f : [a,b] \to \mathbb{R}$, continuous.

$\varphi : J \longrightarrow \mathbb{R}$, differentiable

$J \subseteq [a,b]$. Then for a fixed x_0

$$\frac{d}{dx}\left[\int_{x_0}^{\varphi(x)} f\right] = f(\varphi(x))\,\varphi'(x)$$

Proof

Define $G(x) = \int_{x_0}^{\varphi(x)} f$ and

$$F(x) = \int_{x_0}^{x} f$$

Then

$$G = F(\varphi(x))$$

University of Idaho

$$\frac{d}{dx}\int_{x_0}^{\varphi(x)} f = G'(x) \overset{\text{Chain Rule}}{=} F'(\varphi(x))\,\varphi'(x)$$

$$\overset{\text{FTOC}}{=} f(\varphi(x))\,\varphi'(x)$$

$\square$

University of Idaho

Example: $\frac{d}{dx}\left[\int_0^{\sin x} \cos t \, dt\right]$

direct calculation

$$= \frac{d}{dx}\left[\sin t \Big|_0^{\sin x} \right]$$

$$= \frac{d}{dx}\left[\sin(\sin(x)) \right]$$

$$= \cos(\sin(x)) \cos x .$$

By the corollary to the FTOC

$f(t) = \cos t$, $\varphi(x) = \sin x$

$\varphi' = \cos x$

$$\frac{d}{dx}\left[\int_0^{\sin x} \cos t \, dt\right] = \cos(\sin x)\cos x$$

9

University of Idaho **Problem**

$f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

Define $G(x) = \int_0^x (x-t) f(t)\, dt$

Prove that $G''(x) = f(x)$

University of Idaho

Write G as

$$G(x) = \int_0^x x\,f(t)\,dt - \int_0^x t\,f(t)\,dt$$

$$= x\int_0^x f(t)\,dt - \int_0^x t\,f(t)\,dt$$

$G'(x) =$ Prod. rule + FTOC FTOC

$$\int_0^x f(t)\,dt + x\,f(x) - x\,f(x)$$

$$= \int_0^x f(t)\,dt$$

11

University of Idaho

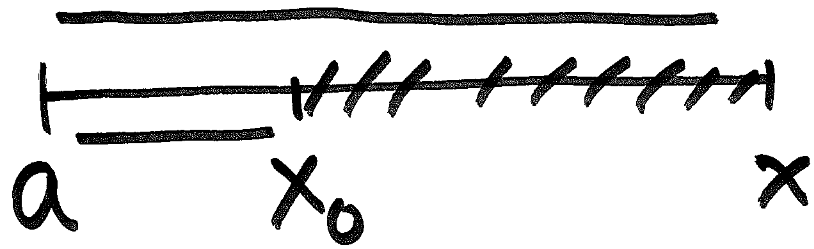
$$G''(x) = \frac{d}{dx}\left[\int_0^x f(t)\,dt\right]$$

$$= f(x) \quad \text{FTOC}$$

$\square$

Scratch

University of Idaho



$$\int_{x_0}^{x} = \int_{x_0}^{a} + \int_{a}^{x} \quad \checkmark$$

$$= -\int_{a}^{x_0} + \int_{a}^{x} \quad \text{obvious}$$