MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 37

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Approximation by Taylor polynomials

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Taylor polynomial

$f : [a,b] \longrightarrow \mathbb{R}$, $x_0 \in (a,b)$

f is differentiable n times, $n \in \mathbb{N}$.

The Taylor polynomial about x_0

$$p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

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Examples

1) $f(x) = e^{x}$, $x_0 = 0$

$$p_n(x) = \underbrace{f(0)}_{1} + \underbrace{f'(0)}_{1} x + \frac{\overbrace{f''(0)}^{1}}{2!} x^2 + \cdots + \frac{\overbrace{f^{(n)}(0)}^{1}}{n!} x^n$$

$f(0) = 1$, $f'(x) = e^{x}$, $f'(0) = 1$

$f''(0) = 1$, $\dots$, $f^{(n)}(0) = 1$

$$p_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

2) $f(x) = \sin x \quad , \quad x_0 = 0$

$$p_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$f(0) = \sin(0) = 0$

$f'(x) = \cos(x) \; ; \; f'(0) = \cos(0) = 1$

$f''(x) = -\sin(x) \; ; \; f''(0) = \sin(0) = 0$

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$$f'''(x) = -\cos(x) \; ; \; f'''(0) = -\cos(0) = -1$$

$\vdots$

$$f^{(2n+1)}(x) = (-1)^{n} \cos x \; ; \; f^{(2n)}(0) = 0$$

$$f^{(2n+1)}(0) = (-1)^{n}$$

$$p_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n} x^{2n+1}}{(2n+1)!}$$

Observation:

$$p_n(x_0) = f(x_0)$$

$$p_n'(x_0) = f'(x_0)$$

$$p_n''(x_0) = f''(x_0)$$

$$\vdots$$

$$p_n^{(n)}(x_0) = f^{(n)}(x_0).$$

Recall:

- Mean Value Thm

$f : [a,b] \longrightarrow \mathbb{R}$

f is continuous on $[a,b]$

f is differentiable on (a,b)

Then $\exists\ c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

- Generalized/Cauchy's Mean Value Thm

[proof done in an earlier lecture)

$f, g : [a, b] \longrightarrow \mathbb{R}$

f, g are continuous on $[a, b]$

f, g are differentiable on (a, b)

$g'(x) \neq 0 \quad \forall \; x \in (a, b)$.

Then $\exists \; c \in (a, b)$ s.t.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Consequence of the MVT

$f : (a,b) \longrightarrow \mathbb{R}$, f has n derivatives. Suppose that at some $x_0 \in (a,b)$

$$f^{(k)}(x_0) = 0, \quad 0 \le k \le n-1$$

Then for each $x \neq x_0$, $\exists$ c strictly between x & x_0 s.t.

$$f(x) = \frac{f^{(n)}(c)}{n!}(x - x_0)^n$$

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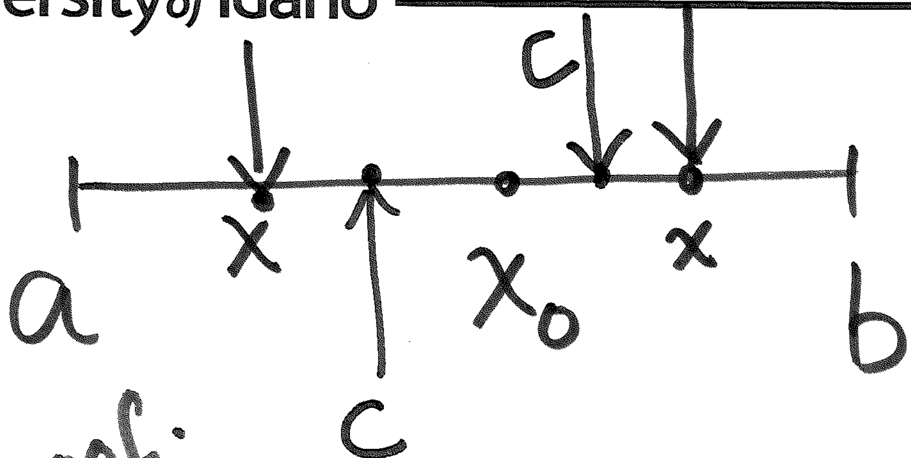

Idea of the proof:

This is proved using the generalized MVT for $g(x) = (x-x_0)^n$ and f.

Apply the MVT n times.

Lagrange's Remainder Thm

$f : (a,b) \longrightarrow \mathbb{R}$ has $n+1$ derivatives, $x_0 \in (a,b)$.

Then for each $x \neq x_0$, $\exists$ c strictly between x & x_0, s.t.

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{Taylor poly } P_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}_{\text{remainder}}$$

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k} + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

Proof: $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$

Let $R(x) := f(x) - p_n(x)$

$R(x_0) = 0 = R'(x_0) = \cdots = R^{(n)}(x_0)$

By the Consequence of the MVT

$\exists$ c strictly between x & x_0

s.t.

$$R(x) = \frac{R^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

$$R^{(n+1)}(c) = f^{(n+1)}(c) - 0 = f^{(n+1)}(c).$$

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$$\Rightarrow R(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

$\square$