MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 38

University of Idaho

Problems

1. For $x > 0$, use Taylor's polynomial to prove that

$$1+\frac{x}{2}-\frac{x^2}{8} \le \sqrt{x+1} \le 1+\frac{x}{2}$$

polynomial of degree 2

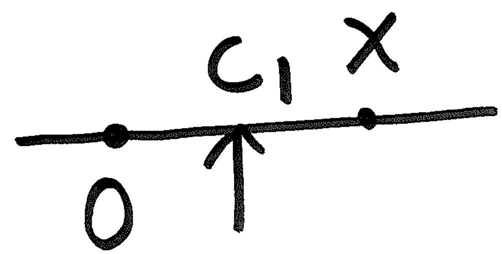
f(x)

polynomial of degree 1

Recall: Lagrange's Remainder Thm

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R_1(x)$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + R_2(x)$$



Let $x_0 = 0$

$$R_1(x) = \frac{f''(c_1)}{2!} x^2 \qquad 0 < c_1 < x$$

3

University of Idaho

$$R_2(x) = \frac{f^{(3)}(c_2)}{3!}x^3 \quad , \quad 0 < c_2 < x$$

$$f(x) = \sqrt{1+x} \; ; \; f'(x) = \frac{1}{2\sqrt{x+1}} \; ;$$

$$f'(0) = \frac{1}{2} \; ; \; f''(x) = \frac{-1}{4(x+1)^{3/2}}$$

$$f''(0) = -\frac{1}{4} \; ; \; f'''(x) = \frac{3}{8(x+1)^{5/2}}$$

$$R_2(x) = \frac{1}{3!}\,\frac{3 \quad x^3}{8(\sqrt{c_2+1})^5} > 0 \text{ since } x > 0$$

4

University of Idaho

$$f(x) = \sqrt{1+x} = f(0) + f'(0)x + R_1(x)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + R_1(x)$$

$R_1 < 0$

$$\sqrt{1+x} = f(0) + x f'(0) + \frac{f''(0)}{2!}x^2 + R_2(x)$$

$$= 1 + \frac{1}{2}x + \left(-\frac{1}{4}\right)\frac{1}{2!}x^2 + R_2(x)$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + R_2(x)$$

$R_2 > 0$

5

$R_2(x) = \frac{1}{3!} \frac{3}{8} \frac{1}{(c_2+1)^{5/2}} x^3 > 0$

$R_1(x) = \frac{-1}{4(c_1+1)^{3/2}} \frac{x^2}{2!} \quad 0 < c_1 < x$

< 0

5(b)

University of Idaho

Since $R_1 < 0$ and $R_2 > 0$

$$1+\frac{x}{2}-\frac{x^2}{8} \underset{<}{\leq} \sqrt{1+x} \underset{<}{\leq} 1+\frac{x}{2}$$

University of Idaho

2. Show that for a natural number n

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + x^n$$

Let $f(x) = (1+x)^n$. This is a polynomial of degree n.

7

$$x_0 = 0$$

University of Idaho

$$f(x) = (1+x)^n \; ; \quad f(0) = 1$$

$$f'(x) = n(1+x)^{n-1} \; ; \quad f'(0) = n$$

$$f''(x) = n(n-1)(1+x)^{n-2} \; ; \quad f''(0) = n(n-1)$$

$$\vdots$$

$$f^{(k)}(x) = n(n-1)\cdots(n-k+1)(x+1)^{n-k}$$

$$f^{(k)}(0) = n(n-1)\cdots(n-k+1)$$

$$0 < k \leq n$$

Taylor polynomial of f at $x_0 = 0$ of degree n is

$$f(0) + f'(0)x + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots + \frac{f^{(n)}}{n!}x^n$$

$$f^{(k)}(0) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

$$\frac{f^{(k)}(0)}{k!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

a

University of Idaho

The nth polynomial is

$\cancel{\$}\ 1 + nx + \cdots + \binom{n}{k} x^k + \cdots + x^n$

Since $(1+x)^n$ is a polynomial of degree n and therefore must equal the nth Taylor polynomial.

$(1+x)^n = 1 + nx + \cdots + \binom{n}{k} x^k + x^n$

Note:

Suppose f is a polynomial of degree at most $\underline{\underline{n}}$. Then the $\underline{\underline{n\text{th}}}$ Taylor polynomial of f at some x_0 is equal to f

$$f(x) = p_n(x)$$

i.e. remainder $R_n = 0$.

N

University of Idaho

$$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n$$

In the formula for $(1+x)^n$ we can use $x = b/a$ to get the expansion for $(a+b)^n$.