# MATH 471

# INTRODUCTION TO ANALYSIS I

## SESSION no. 39

# Improper Integrals

So far:

$f : [a,b] \longrightarrow \mathbb{R}$

$f$ is bounded

$[a,b]$ : domain of $f$ is also bounded

Improper Integral: if $f$ is unbounded or the domain is unbounded: $(a,\infty)$, $(-\infty, b)$ $(-\infty, \infty)$

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Case 1

Unbounded interval:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

[assuming that f is bounded]

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# Example

$$\int_1^\infty x e^{-x} \, dx$$

$$f(x) = x e^{-x}$$

bounded
integrable on
every finite int-
erval $(a, b)$

$$\int_1^\infty x e^{-x} \, dx = \lim_{b \to \infty} \int_1^b x e^{-x}$$

int.
by
parts

$$= \lim_{b \to \infty} \left[ -x e^{-x} \Big|_1^b + \int_1^b e^{-x} \, dx \right]$$

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$$\lim_{b\to\infty}\left[-be^{-b}+e^{-1}-\left[e^{-x}\right]_1^b\right]$$

$$=\lim_{b\to\infty}\left[-\underbrace{\cancel{\frac{b}{e^b}}}_{0}+e^{-1}-\underbrace{\cancel{e^{-b}}}_{0}+e^{-1}\right]$$

$$=2e^{-1}$$

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$$\int_1^\infty \frac{dx}{\sqrt{x}}$$

$f(x) = \dfrac{1}{\sqrt{x}}$ is bounded and continuous on $(1, b)$ and so intergrable on every bounded interval $(1, b)$

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$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}}$$

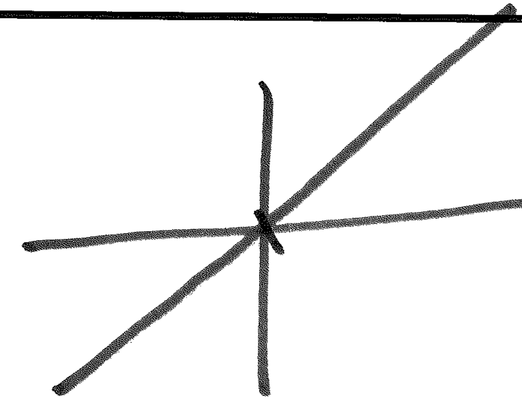
$$= \lim_{b\to\infty} \Big[ 2\sqrt{x} \Big]_1^b$$

$$= \lim_{b\to\infty} 2\sqrt{b} - 2$$

The integral diverges to $\infty$.

# Example

$$\int_{-\infty}^{\infty} x\,dx \;:$$



1) $$\lim_{k\to\infty}\left[\int_{-k}^{k} x\,dx\right]$$

$$= \lim_{k\to\infty}\left[\frac{x^2}{2}\right]_{-k}^{k} = \lim_{k\to\infty}\left[\frac{k^2}{2} - \frac{k^2}{2}\right]$$

$$= 0$$

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2) $$\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$$

$$= \lim_{a \to -\infty} \int_{a}^{0} x\,dx + \lim_{b \to \infty} \int_{0}^{b} x\,dx$$

$$= \lim_{a \to -\infty} \frac{x^2}{2}\Big|_{a}^{0} + \lim_{b \to \infty} \frac{x^2}{2}\Big|_{0}^{b}$$

$$= -\infty + \infty$$

indeterminate

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If **both** $\int_{-\infty}^{a} f$ and $\int_{a}^{\infty} f$ (take some finite no.) **converge** for **some** real $a$,

then $\int_{-\infty}^{\infty} f$ also converges

and we define $\int_{-\infty}^{\infty} f = \int_{-\infty}^{a} f + \int_{a}^{\infty} f$ .

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If $\int_{-\infty}^{\infty} f$ converges as defined

then it equals $\lim_{a \to \infty} \int_{-a}^{a} f(x)\,dx$

This is called Cauchy's principal value.

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$$\int_{-\infty}^{\infty} e^{-x/2}$$

Caution: DO NOT evaluate and plug in $\pm\infty$.

$$\int_{-\infty}^{\infty} e^{-x/2} = -2e^{-x/2}\Big|_{-\infty}^{\infty}$$

$$= -2\left[0 - \infty\right] = \infty$$

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$a=0$

$$\int_{-\infty}^{\infty} e^{-x/2} = \int_{-\infty}^{0} e^{-x/2} dx + \int_{0}^{\infty} e^{-x/2}$$

$$= \lim_{a \to -\infty} \int_{a}^{0} e^{-x/2} + \lim_{b \to \infty} \int_{0}^{b} e^{-x/2}$$

$$= \lim_{a \to -\infty} -2 e^{-x/2} \Big|_{a}^{0} + \lim_{b \to \infty} -2 e^{-x/2} \Big|_{0}^{b}$$

$$= -2 + 2 = 0$$

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$$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \pi$$

Substitute $\quad x+1=y$

$\vdots$

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## Case 2

Interval bounded, function gets unbounded:

$\cancel{\int_0^8 \frac{1}{(8-x)^{4/3}}}$ $\qquad \int_0^1 \frac{dx}{\sqrt{x}}$

$f(x) = \frac{1}{\sqrt{x}}$ is unbounded at $x = 0$.

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$$\lim_{\substack{c \to 0 \\ c > 0}} \int_c^1 \frac{dx}{\sqrt{x}} = \lim_{c \to 0} 2\sqrt{x}\Big|_c^1$$

$$= \lim_{c \to 0} \left[ 2\cancel{\sqrt{1}} - 2\sqrt{c} \right] = 2$$

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$$\int_{-3}^{2} \frac{dx}{x^2}$$

$\frac{1}{x^2}$ is unbounded at $x = 0$

Caution: Do not try to evaluate the integral and plug in $-3$ and $2$.

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$$\left[-x^{-1}\right]_{-3}^{2} = -\frac{1}{2} + \frac{1}{3}$$

$$\int_{-3}^{2}\frac{dx}{x^2} = \int_{-3}^{0}\frac{1}{x^2} + \int_{0}^{2}\frac{1}{x^2}$$

$$= \cancel{-\frac{1}{x}\Big|_{-3}^{0}}\ \lim_{a\to 0}\int_{-3}^{a}\frac{1}{x^2} + \lim_{c\to 0}\int_{c}^{2}\frac{1}{x^2}$$

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$$= \lim_{\substack{a\to 0 \\ a<0}} -\frac{1}{x}\Bigg|_{-3}^{a} + \lim_{\substack{c\to 0 \\ c>0}} -\frac{1}{x}\Bigg|_{c}^{2}$$

$$= \underbrace{\lim_{a\to 0}\left[\frac{1}{a} - \frac{1}{3}\right]}_{\text{diverges}} + \underbrace{\lim_{c\to 0}\left[-\frac{1}{2} + \frac{1}{c}\right]}_{\text{diverges}}$$