MATH 471

INTRODUCTION TO ANALYSIS I

SESSION no. 40

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5 Find the Taylor polynomial at $\underline{x=1}$ for $f = x^5 - x^3 + x$

$$p_5(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5$$

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$f(1) = 1$, $f'(1) = 3$

$f''(1) = 14$, $f^{(3)}(1) = 54$, $f^{(4)}(1) = 120$

$f^{(5)}(1) = 120$

$$p_5(x) = 1 + 3\underline{(x-1)} + \frac{14}{2!}\underline{(x-1)}^2 +$$

$$\frac{54}{3!}\underline{(x-1)}^3 + \frac{120}{4!}\underline{(x-1)}^4 + \frac{120}{5!}\underline{(x-1)}^5$$

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At $x=0$, the Taylor polynomial for f is the same as itself:

$$x^5 - x^3 + x$$

4) See the class-notes.

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#3 $f : \mathbb{R} \to \mathbb{R}$

Given: $f''(x) + f(x) = e^{-x}$

$f(0) = 0, \ f'(0) = 2$

Find the 4th polynomial @ $x_0 = 0$.

$f''(x) = e^{-x} - f(x) \ ; \ f''(0) = 1$

$f'''(x) = -e^{-x} - f'(x) ; \ f'''(0) = -3$

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$$f^{(4)}(x) = e^{-x} - f''(x) = 0$$

$$p_4 = \cancel{f(0)} + f'(0)x + \frac{f''(0)}{2!}x^2$$

0

$$+ \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$= 2x + \frac{1}{2}x^2 - \frac{3}{3!}x^3$$

$$= 2x + \frac{x^2}{2} - \frac{x^3}{2}$$

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#2 Compute the third polynomial for

$$f(x) = \int_0^x \frac{1}{1+t^2}\,dt$$

about $x_0 = 0$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

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$$f(0) = \int_0^0 \frac{1}{1+t^2}\,dt = 0$$

$$f'(0) = \underline{1} \quad ; \quad f'(x) = \frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt$$

$$= \frac{1}{1+x^2}$$

$$f''(x) = \frac{-1(2x)}{(1+x^2)^2} \quad ; \quad f''(0) = 0$$

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$$f'''(x) = \frac{-2}{(1+x^2)^2} + \frac{(-2x)\ \ 2x}{(1+x^2)^3}(-2)$$

$$f'''(0) = -2$$

$$p_3(x) = x - \frac{2}{3!}x^3 = x - \frac{x^3}{3}$$

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Comments on FTOC:

$$f(x) = \int_0^{x^3} \frac{1}{1+t}\,dt$$

$$f'(x) = \frac{1}{1+x^3}\left(3x^2\right)$$

Recall: $\displaystyle\int_a^{\varphi(x)} f(t)\,dt = F(x)$

$F'(x) = f(\varphi(x))\,\varphi'(x)$

a ← any const.

See lecture notes

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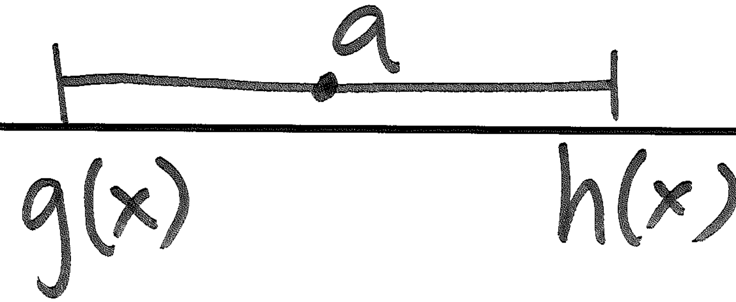
$$F(x) = \int_{g(x)}^{h(x)} f(t)\,dt$$



$$F'(x) = \int_{g(x)}^{a} + \int_{a}^{h(x)}$$

$$= -\int_{a}^{g(x)} + \int_{a}^{h(x)} \quad \xrightarrow[\text{or its corollary}]{\text{FTOC}}$$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\int_{x_0}^{x_1} f''(x)\,dx = f'(x_1) - f''(x_0)$$

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#1 (a) $f(x) = \ln(x)$

$\underline{\underline{n}}$ th polynomial about $x_0 = 1$.

$$P_n = f(1) + f'(1)(x-1) + \frac{f^{(2)}(1)}{2!}(x-1)^2$$

$$+ \frac{f^{(3)}(1)}{3!}(x-1)^3 + \cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n$$

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$f(1) = \ln(1) = 0$

$f'(x) = \frac{1}{x}$ $\qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $\qquad f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(1) = 2$

$f^{(4)}(x) = \frac{-2(3)}{x^4}$ $\qquad f^{(4)}(1) = -3!$

$\vdots$

$f^{(n)}(1) = (-1)^{n+1}(n-1)!$

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$$p_n(x) = (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!}\,2$$

$$+ \cdots + \frac{(x-1)^n}{n!}(n-1)!\;(-1)^{n+1}$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + \frac{(x-1)^n}{n}(-1)^{n+1}$$

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1(b) $f(x) = \frac{1}{1-x}$ about $x_0 = 0$

$p_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$.