MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 1

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Infinite Series

Consider a sequence of nos.

$$\{a_n\}_{n=1}^{\infty}, \quad a_n \in \mathbb{R}.$$

Construct a new sequence $\{s_n\}_{n=1}^{\infty}$:

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$s_n = a_1 + a_2 + \cdots + a_n$$

s_n is called the <u>n th partial sum</u>

The sum

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

$$= \sum_{k=1}^{\infty} a_k$$

is called an infinite series.

The sequence $\{s_n\}_{n=1}^{\infty}$ is called the sequence of partial sums of the (infinite) series $\sum_{k=1}^{\infty} a_k$.

Note: $s_n = s_{n-1} + a_n$

(recurcive relation)

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Convergence of series

Example 1: $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ Geometric Series

$$= \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)$$

$$= \frac{1}{2}\,\frac{1}{1-\frac{1}{2}} = 1$$

The series converges.

Example 2

$$\sum_{k=1}^{\infty} \frac{1}{k}$$ Harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty, \text{ not finite}$$

The sequence does not converge, it is said to diverge.

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Definition (Convergence)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

The series $\sum_{k=1}^{\infty} a_k$ converges to some s if and only if

$$\lim_{n \to \infty} s_n = s$$

If $\{s_n\}_{n=1}^{\infty}$ diverges, then the series $\sum_{k=1}^{\infty} a_k$ is said to diverge.

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$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n$$

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Convergence of the geometric series

Proposition: For a number r, such that $|r|<1$,

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

Proof:

$$s_n = 1 + r + r^2 + \cdots + r^n$$

$$r s_n = r + r^2 + \cdots + r^n + r^{n+1}$$

Subtract: $(1-r)s_n = 1 - r^{n+1}$

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From 471

$$\lim_{n\to\infty} r^n = 0 \text{ if } |r| < 1$$

$$\Rightarrow \quad s_n = \frac{1 - \overset{0}{\cancel{r^{n+1}}}}{1-r}$$

$$\sum_{k=1}^{\infty} r^k \underset{\text{by definition}}{=} \lim_{n\to\infty} s_n = \frac{1}{1-r}$$

$\square$

$$s_n = \underbrace{\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1}}_{n \text{ terms each } \geqslant \frac{1}{2}} \geqslant \frac{n}{2}$$

$\to$ non-negative

Compare $\{s_n\}$ with $\left\{\frac{n}{2}\right\}$

$\left\{\frac{n}{2}\right\}$ is divergent and terms of $\{s_n\}$ are $\geqslant$ terms of $\left\{\frac{n}{2}\right\}$

$\Rightarrow \{s_n\}$ diverges and by the definition the series $\sum_{k=1}^{\infty} \frac{k}{k+1}$ diverges.

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Example (divergence)

Prove that $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k+1}$ diverges.

nth term is $\dfrac{n}{n+1} = \dfrac{1}{1+\frac{1}{n}}$

increases with n.

$n=1$ gives $\dfrac{1}{1+1} = \dfrac{1}{2}$

Thus $\dfrac{n}{n+1} \geqslant \dfrac{1}{2}$ for $n = 1, 2, 3, \ldots$