MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 2

Convergence of Series

Proposition: Suppose that

$$\sum_{n=1}^{\infty} a_n \quad \text{converges.}$$

Then $$\lim_{n \to \infty} a_n = 0$$

Proof: Let $\{s_n\}$ be the sequence of partial sum of $\sum_{n=1}^{\infty} a_n$.

Since $\sum a_n$ converges

$\lim_{n \to \infty} s_n = s$, for some s.

Note that $\lim_{n \to \infty} s_{n-1} = s$.

$s_n = s_{n-1} + a_n \Rightarrow s_n - s_{n-1} = a_n$.

But $\lim_{n \to \infty} [s_n - s_{n-1}] = 0$

$\Rightarrow \lim_{n \to \infty} a_n = 0$

$\square$

Remarks

1) The converse of the above result is not true

$$\sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges} \; ; \; a_n = \frac{1}{n}$$

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

2) If $a_n \not\to 0$, then the result says that the series must diverge.

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3) This result must be used to prove divergence <u>NOT</u> convergence.

If $\{a_k\}$ and $\{b_k\}$ are two sequence then

$$\sum_{k=1}^{n}(\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k;$$

if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent then so is $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$

and $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{\infty} a_k + \beta \sum_{k=1}^{\infty} b_k.$

Tests for convergence of series

(for positive series)

1. Theorem: Suppose $\{a_k\}_{k=1}^{\infty}$ is a seq. of non-negative nos. Then

$$\sum_{k=1}^{\infty} a_k \text{ converges } \iff$$

$\exists\, M$ s.t.

$$\forall n, \quad \underbrace{a_1 + \cdots + a_n}_{s_n} \le M$$

i.e. the sequence of partial sums is bounded.

Proof: The terms of $\sum_{k=1}^{\infty} a_k$ being non-negative, the seq. $\{s_n\}$ is monotonically increasing:

$$s_n = s_{n-1} + \overset{\text{non neg}}{a_n} \geqslant s_{n-1}$$

$\{s_n\}$ is monotone increasing and bounded above by M. By the Monotone Convergence Theorem

$\Longleftrightarrow$

this is true if and only if $\{s_n\}$ is convergent that is the series $\sum_{k=1}^{\infty} a_k$ converges

$\square$

2) The Comparison Test: Suppose $\{a_k\}$ and $\{b_k\}$ are seq.s. s.t.

$\forall k \quad 0 \leq a_k \leq b_k$

(i) The series $\sum_{k=1}^{\infty} a_k$ converges if $\sum_{k=1}^{\infty} b_k$ converges

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} b_k$ also diverges.

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Proof of (1)

We have $\sum_{k=1}^{n} a_k \le \sum_{k=1}^{n} b_k$

Since $\sum_{k=1}^{\infty} b_k$ converges, by the previous thm, $\sum_{k=1}^{n} b_k$ are bounded for each n.

$\Rightarrow \sum_{k=1}^{n} a_k$ are bounded for all n

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$\Rightarrow$ by the previous thm (applied to $\sum a_k$) that

$$\sum_{k=1}^{\infty} a_k \text{ converges.}$$

Examples (Comparison Test)

1) Consider $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$

$$\frac{1}{\sqrt{k}} \geq \frac{1}{k} \qquad k = 1, 2, \ldots$$

We know that $\sum \frac{1}{k}$ diverges and so by the Comparison test $\sum \frac{1}{\sqrt{k}}$ diverges

2) Consider $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\, 2^k}$

For each k, $\frac{1}{\sqrt{k}\, 2^k} \leq \frac{1}{2^k}$

We know that $\sum_{k=1}^{\infty} \frac{1}{2^k}$

is the geometric series and

it converges. Therefore (by the Comparison Test)

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\,2^k}$$ converges.