MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 3

Previously: Comparison Test

Today: Integral Test

Both are tests for non-negative series

Integral Test

Let $\{a_k\}_{k=1}^{\infty}$ is a non-neg seq of nos. and let $f:[1,\infty) \to \mathbb{R}$ be continuous and monotone decreasing s.t. $f(k) = a_k, \forall k \in \mathbb{N}$,

Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

Example (Integral Test)

Prove the convergence or divergence of $\sum_{k=1}^{\infty} \frac{1}{(k+1)\ln(k+1)}$

$a_k = \frac{1}{(k+1)\ln(k+1)}$

Let $f(x) = \frac{1}{(x+1)\ln(x+1)}$; $f(k) = a_k$

$\longrightarrow$ continuous & monotone

$$\int_1^{\infty} \frac{1}{(x+1)\ln(x+1)}\, dx$$

$$y = \ln(x+1) \Rightarrow dy = \frac{dx}{(x+1)}$$

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$$\int_{\ln 2}^{\infty} \frac{dy}{y} = \ln \infty - \ln(\ln 2)$$
$$= \infty$$

By the integral test, the given series diverges.

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Proof of the integral test:

$$f(k+1) \le f(x) \le f(k), \quad \forall x \in [k, k+1]$$

$$\forall x \in [k, k+1]$$

f continuous $\Rightarrow$ f is integrable

$$\int_k^{k+1} f(k+1) \le \int_k^{k+1} f(x)\,dx \le \int_k^{k+1} f(k)\,dx$$

$$f(k+1) \le \int_k^{k+1} f(x)\,dx \le f(k)$$

$$a_{k+1} \le \int_k^{k+1} f(x)\,dx \le a_k$$

Add for $k=1,\dots,n$

$$a_2 + a_3 + \cdots + a_{n+1} \le \int_1^2 + \int_2^3 + \cdots + \int_n^{n+1}$$

$$\le a_1 + a_2 + \cdots + a_n$$

$$\Rightarrow s_{n+1} - a_1 \le \int_1^{n+1} f(x)\,dx \le s_n$$

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Assume that $\sum_{k=1}^{\infty} a_k$ converges.

By a theorem in Lecture 2:

the partial sums s_n are bounded

i.e. $\exists$ M, s.t.

$$\int_1^{n+1} f(x)dx \leq s_n \leq M \quad \forall n$$

$$\int_1^\infty f(x)\,dx = \lim_{n\to\infty} \int_1^{n+1} f(x)\,dx$$

The sequence $\left\{ \int_1^n f(x)\,dx \right\}_{n=1}^{\infty}$

is increasing and bounded; by the MCT it converges

$\Longrightarrow \int_1^\infty f(x)\,dx$ converges.

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Conversely, assume that $\int_1^\infty f(x)\,dx$ is convergent. We know

$$s_{n+1} - a_1 \le \int_1^{n+1} f(x)\,dx$$

$$\Rightarrow s_{n+1} \le \int_1^{n+1} f(x)\,dx + a_1$$

$$\le \underbrace{\int_1^\infty f(x)\,dx + a_1}_{\text{independent of } n}$$

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Thus $\{s_{n+1}\}$ is a bounded seq;
$\{s_{n+1}\}$ is also an increasing seq
(because the a_ns are non-neg)
By the MCT $\{s_n\}$ is
convergent $\Longrightarrow \sum_{n=1}^{\infty} a_n$ is
convergent $\square$

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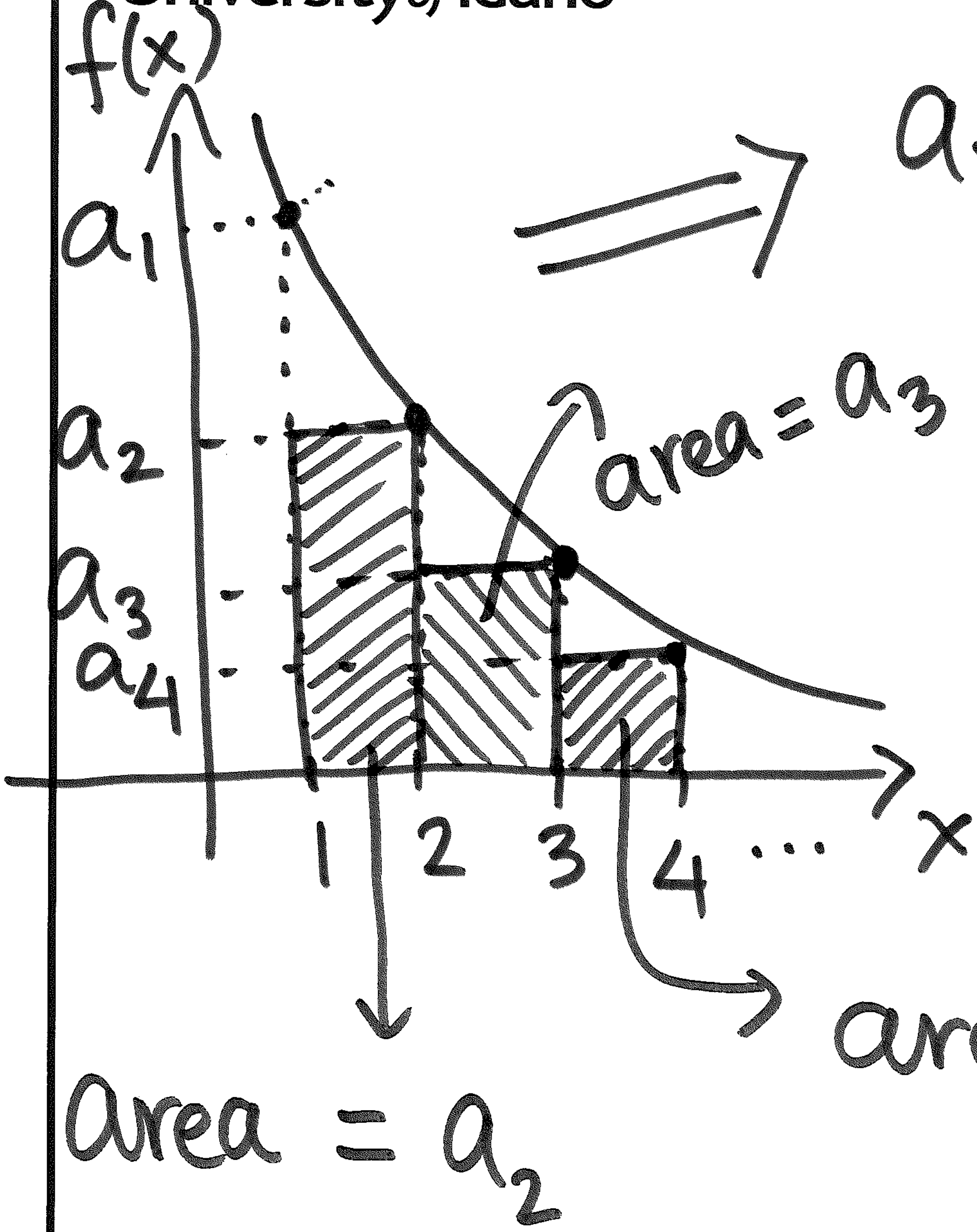

$$\Longrightarrow a_2 + a_3 + a_4 + \cdots \le \int_1^\infty f(x)\,dx$$

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$$\int_1^\infty f(x)\,dx \le a_1 + a_2 + a_3 + \cdots$$