MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 4

Last lecture : Integral Test

The p-test : For a positive no. p, the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges & if and only if $p > 1$.

[Corollary to the Integral test]

Proof of the p-test.

Let $f(x) = \frac{1}{x^p}$. $f(k) = \frac{1}{k^p}$.

$f(x)$ is continuous & monotone decreasing.

$$\int_1^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_1^{b} f(x)\,dx$$

$$= \lim_{b \to \infty} \int_1^{\infty} \frac{1}{x^p}\,dx$$

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$p=1$: $\int_1^\infty \frac{1}{x}\,dx = \ln x \Big|_1^\infty = \infty$

$\Rightarrow$ The series diverges for $p=1$

$\Rightarrow \sum_{k=1}^\infty \frac{1}{k}$ diverges.

If $p \neq 1$:

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$$

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$$= \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^b$$

$$= \lim_{b\to\infty} \left[\frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1} \right]$$

$$b^{-p+1} = \frac{1}{b^{p-1}}$$

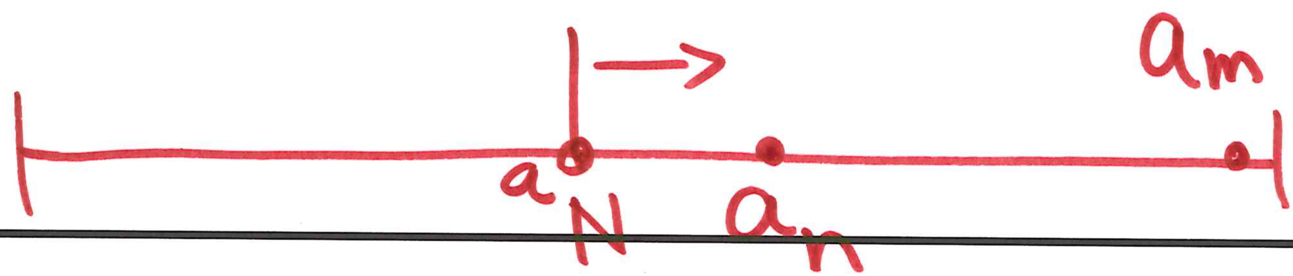
$$= \begin{cases} -\frac{1}{-p+1} & p>1 \\ \infty & 0<p<1 \end{cases}$$

5. Convergence of series (not just positive series)

Cauchy sequence:

A sequence $\{a_n\}$ is Cauchy if given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \quad \text{for } \begin{array}{l} n \geq N \\ m \geq N \end{array}$$



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Theorem

Every convergent sequence is Cauchy.

<u>Proof</u>: Suppose $\{a_n\}$ converges to a. Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$|a_n - a| < \varepsilon, \quad n \geq N.$$

to be found

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Want to show: given ε, $\exists$ $N \in \mathbb{N}$ s.t.

$$|a_m - a_n| < \varepsilon, \quad m, n \geq N.$$

exists due to conv.

By convergence, for a given ε, $\exists\, N^* \in \mathbb{N}$ s.t.

$$|a_k - a| < \frac{\varepsilon}{2}, \quad k \geq N^*.$$

$$|a_m - a_n| = |\overbrace{a_m - a} + \overbrace{a - a_n}|$$

$$\leq |a_m - a| + |a - a_n|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

when $m, n \geq N^*$

Pick $N = N^*$ to get the Cauchy condition $\square$

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Theorem: Cauchy criteria for series

A series $\sum_{k=1}^{\infty} a_k$ converges

$\iff$ for each $\varepsilon > 0$ $\exists N \in \mathbb{N}$

s.t.

$$|a_m + a_{m+1} + \cdots + a_n| = \left|\sum_{k=m}^{n} a_k\right| < \varepsilon$$

whenever $n \geq m \geq N$.

Cauchy criteria

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$$\sum_{k=1}^{\infty} a_k \text{ converges}$$

partial sums

$$\iff \{s_n\} \text{ converges}$$

$$\iff \{s_n\} \text{ is a Cauchy seq.}$$

$$\iff |s_n - s_{m-1}| < \varepsilon, \quad n \geq m-1 \geq N \in \mathbb{N}$$

$$\iff |a_m + a_{m+1} + \cdots + a_n| < \varepsilon, \quad n \geq m \geq N \in \mathbb{N} \quad \square$$