MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 5

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1b) $$\sum \frac{k}{e^k}$$

Try the integral test

[Can use the comparison test, might be harder]

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Convergence of series

[not necessarily non-negative]

3 Corollary to the Cauchy conv. criteria:

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(see Lecture 4)

If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then so does $\sum_{k=1}^{\infty} a_k$.

Proof: By the Cauchy conv. criteria

$$\big|\, |a_m| + |a_{m+1}| + \cdots + |a_n| \,\big| < \varepsilon, \quad m, n \geq N \in \mathbb{N}$$

$$\Rightarrow |a_m| + |a_{m+1}| + \cdots + |a_n| < \varepsilon, \quad m, n \geq N$$

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$|a_m + a_{m+1} + \cdots + a_n| \leq$ by Triangle

$|a_m| + |a_{m+1}| + \cdots + |a_n| < \varepsilon$

$\forall m, n \geq N \in \mathbb{N}$

$\Rightarrow |a_m + \cdots + a_n| < \varepsilon, \; m, n \geq N$

$\Rightarrow$ by Cauchy Criteria the series $\sum a_k$ converges. $\square$

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Does the converse hold?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges.}$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

Thus, the converse does not hold.

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Absolute Convergence

The series $\sum_{k=1}^{\infty} a_k$ is said to converge absolutely if the series $\sum_{k=1}^{\infty} |a_k|$ converges

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Example

Show that $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ converges.

$$\sum_{k=1}^{\infty} \left| \frac{\sin k}{k^2} \right| \leq \sum_{k=1}^{\infty} \frac{1}{k^2}$$ which

converges by the p-test ($p = 2 > 1$)

By the comparison test $\sum \left| \frac{\sin k}{k^2} \right|$ also converges.

$\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ is thus absolutely convergent $\Rightarrow$ by the Corollary that $\sum \frac{\sin k}{k^2}$ converges.

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Alternating series Test

If $\{a_k\}_{k=1}^{\infty}$ is a sequence of non-negative nos. such that

(1) $a_{k+1} \le a_k \quad \forall k$

(2) $\lim_{k \to \infty} a_k = 0$

then $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ Converges

[Proof in lecture 6]

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$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$a_n = \frac{1}{n} > 0$$

$$a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$$

By the alt. series test

$\sum \frac{(-1)^{n+1}}{n}$ converges.

Remark:

There can be a convergent alternating series such that $a_{k+1} \le a_k$ is <u>not</u> true.

Consider $\sum (-1)^{k+1} a_k$ where

$$a_k = \begin{cases} \frac{1}{k^2} & \text{if } k \text{ is even} \\ \frac{1}{k^3} & \text{if } k \text{ is odd} \end{cases}$$

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$$\frac{1}{2^2} \quad \frac{1}{3^3} \quad \frac{1}{4^2}$$

$$\frac{1}{4} \geqslant \frac{1}{27} \leq \frac{1}{16}$$

$$a_2 \quad a_3 \quad a_4$$

$a_{k+1} \leq a_k$ <u>does not hold</u>

But $\sum (-1)^{k+1} a_k$ converges.

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$$\left.\begin{array}{l}\frac{1}{k^2}\\ \frac{1}{k^3}\end{array}\right\} \le \frac{1}{k^p} = \frac{1}{k^{3/2}} \qquad p = 3/2$$

By the p-test $\sum \frac{1}{k^{3/2}}$ converges, and by absolute convergence & the comparison test $\sum (-1)^k a_k$ converges.