MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 6

Alternating Series Test

If $\underline{a_n \geq 0}$; $\underline{a_{n+1} \leq a_n}$; $a_n \to 0$

then $$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$ converges

<u>Proof</u>: $$s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$$

Look at the convergence of the subsequence $\{s_{2n}\}_{n=1}^{\infty}$

$$s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0$$

$s_{2n+2} \geq s_{2n} \Rightarrow$ ① $\{s_{2n}\}$ is monotonically increasing sequence.

② $$s_{2n} = \sum_{k=1}^{n} (a_{2k-1} - a_{2k})$$

$$= a_1 - \sum_{k=1}^{n-1} (a_{2k} - a_{2k+1}) - a_{2n}$$

$$\leq a_1$$

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By the Monotone Conv. Thm

$\{s_{2n}\}_{n=1}^{\infty}$ converges. Let

$$\lim_{n\to\infty} s_{2n} = s$$

$$s_{2n+1} = \underbrace{s_{2n}}_{\to s} + \underbrace{a_{2n+1}}_{\to 0}, \forall n$$

$$\lim_{n\to\infty} s_{2n+1} = s$$

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Claim: $\{s_n\}_{n=1}^{\infty}$ converges to s.

Given $\varepsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ s.t.

$$\begin{cases} |s_{2n} - s| < \varepsilon, & n \geq N_1 \\ |s_{2n+1} - s| < \varepsilon, & n \geq N_2 \end{cases}$$

Let $N = \max\{2N_1, 2N_2+1\}$

Then $|s_n - s| < \varepsilon$, $n \geq N$

Thus $\{s_n\}_{n=1}^{\infty}$ converges

to $s \implies \sum_{k=1}^{\infty} (-1)^{k+1} a_k$

also converges $\square$

Ratio & root tests

Ratio Test : For $\sum_{k=1}^{\infty} a_k$,

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \ell$$

(i) If $\ell < 1$, then $\sum_{k=1}^{\infty} a_k$ converges absolutely $\longrightarrow \sum |a_k| < \infty$

(ii) If $\ell > 1$, the series diverges

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$$-(k+1)^2 + k^2 = -2k-1$$

$$\sum_{k=1}^{\infty} k e^{-k^2}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)\, e^{-(k+1)^2}}{k e^{-k^2}}\right|$$

$$= \lim_{k\to\infty}\frac{k+1}{k} \quad e^{-2k-1} = 0 < 1$$

$\Rightarrow$ the series converges

$$\underbrace{\frac{k+1}{k}}_{\to 1} \quad \frac{1}{e^{2k+1}} \longrightarrow 0$$

$$\sum \frac{(-1)^{n+1}}{n}$$

$$a_n = \frac{1}{n} \longrightarrow 0$$

$$> 0$$

$$\frac{1}{n} > \frac{1}{n+1}$$