MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 7

Ratio & root test

Example from the last lecture

$$\sum_{k=1}^{\infty} k e^{-k^2}$$

proved convergence using the Ratio Test

Alternative test: the Integral Test

$$\int_1^{\infty} x e^{-x^2} dx$$

Substitute: $u = x^2$

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Root test

For $\sum_{k=1}^{\infty} a_k$, let

$$\lim_{n \to \infty} |a_n|^{1/n} = \ell.$$

1) If $\ell < 1$, the series converges absolutely

2) If $\ell > 1$, the series diverges

Remark :

Both the ratio & the root test are inconclusive when $\ell = 1$.

Example : $\displaystyle\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{k}$

Root test : $\displaystyle\lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^{n}\right]^{1/n}$

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$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$$

Ratio Test : $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} = \frac{e}{e} = 1$$

Both ratio & root tests are inconclusive.

An important result:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

For $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$$

$\Rightarrow$ The series diverges.

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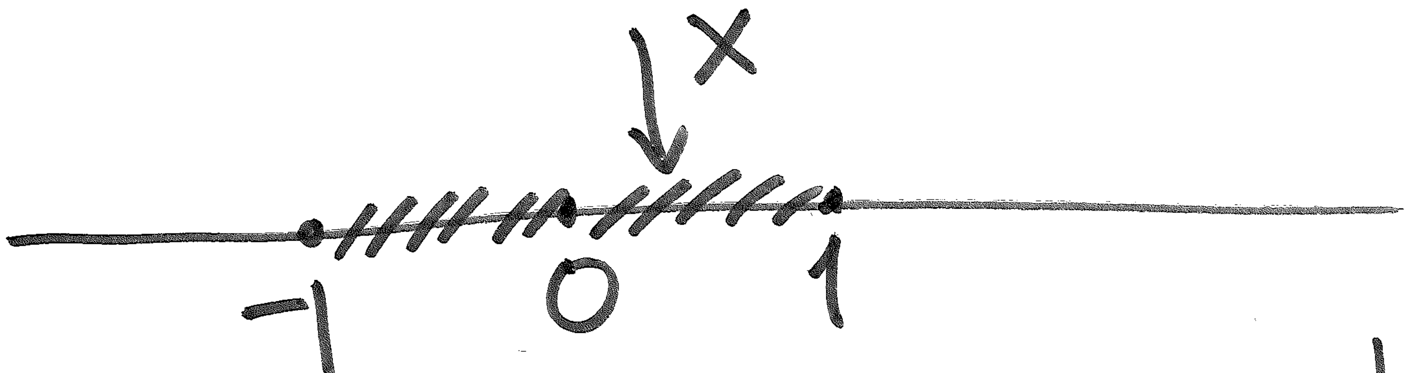
Find the values of x for which $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges

Ratio Test: $\lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)} \frac{n}{x^n} \right|$

$= |x| \lim_{n \to \infty} \left| \underbrace{\frac{n}{n+1}}_{\to 1} \right| = |x|$

If $|x| < 1$, the series Converges



$x = \pm 1$ then the ratio is $|x| = 1$, the ratio test is inconclusive.

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$x = 1 \Longrightarrow$ the series is $\sum_{n=1}^{\infty} \frac{1}{n}$

and diverges

$x = -1 \Longrightarrow$ the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

which converges by the alternating series test

For convergence $x \in [-1, 1)$

Example

$$\sum_{k=1}^{\infty} \frac{3^{2k+1}}{k^{2k}} \qquad a_k = \frac{3^{2k+1}}{k^{2k}}$$

$$\sqrt[k]{a_k} = \left(\frac{3^{2k+1}}{k^{2k}}\right)^{1/k} = \frac{3^2\,3^{1/k}}{k^2}$$

$$\lim_{k\to\infty} a_k^{1/k} = \lim_{k\to\infty} \frac{3^2\,3^{1/k}}{k^2} = 0 < 1$$

By the root test, we have convergence.

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Sequences and series of functions

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Let $f_n(x) = x^n$ $\longrightarrow$ $n=1 \Rightarrow f_1 = x$

$n \in \mathbb{N}$

$$\{f_n\}_{n=1}^{\infty} = \{x^n\}_{n=1}^{\infty} = \{1, x, x^2, \cdots\}$$

$0 \le x \le 1$

(a) $x = 1$: $\underbrace{\{1, 1, 1, \ldots\}}_{\{f_n(1)\}} \longrightarrow 1$

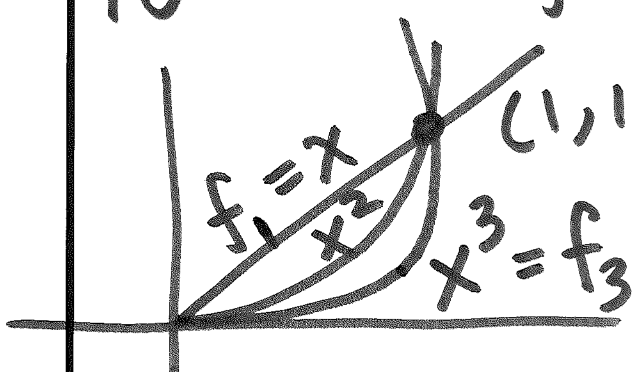
$x \in [0,1)$

$\{f_n\} = \{x^n\} \longrightarrow 0$

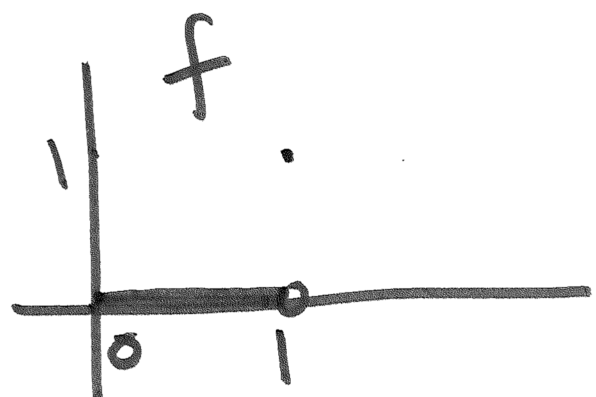
$\longrightarrow$ all continuous funcs.

$\{f_n\}$ converges pointwise to a function

$$f = \begin{cases} 1 \; ; & x = 1 \\ 0 \; ; & 0 \le x < 1 \end{cases}$$



$\longrightarrow$



↑ limit is NOT continuous