MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 8

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Pointwise convergence

Given a seq. of functions $\{f_n\}$ we say that the seq $\{f_n\}$ **converges pointwise** to f if

$$\lim_{n \to \infty} f_n(x) = f(x), \quad \forall x$$

in the domain of the f_ns & f

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pointwise conv. :

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i.e. given ε, $\exists\, N$ such that for each x

$$|f_n(x) - f(x)| < \varepsilon, \quad n \geq N$$

$N = N(\varepsilon, x)$ dependence on both ε, x.

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$f_n : [a, b] \longrightarrow \mathbb{R}$

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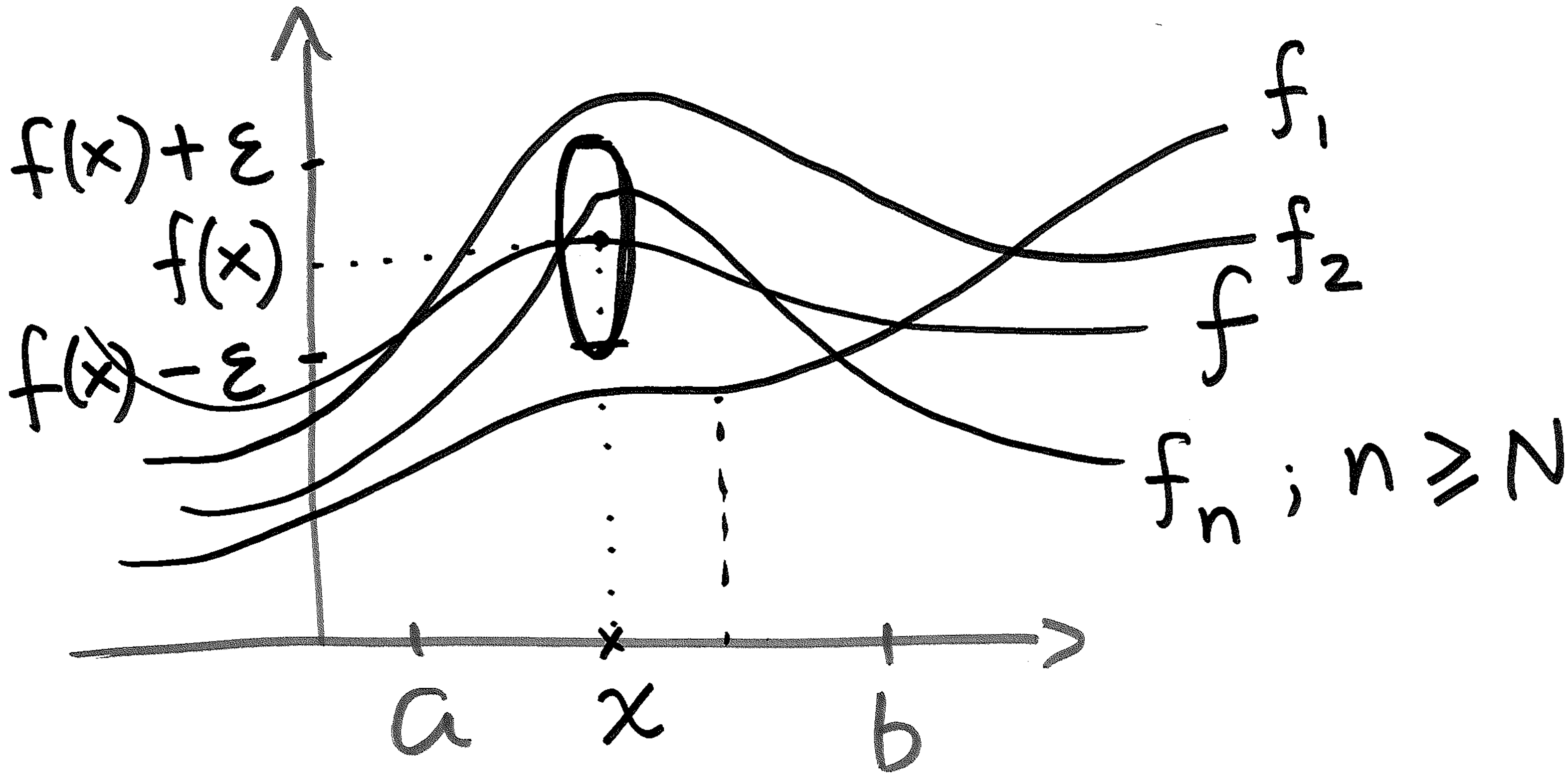

$f_n \xrightarrow[\text{pointwise}]{} f$ $\quad N(\varepsilon, x)$

$f : [a, b] \longrightarrow \mathbb{R}$

From the last lecture

$$\{f_n(x) = x^n\} \xrightarrow[\text{pointwise}]{} f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & x = 1 \end{cases}$$

Each x^n is continuous but f is not continuous

Each x^n is differentiable but f is not differentiable

$$f_n(x) = n x^n \qquad f_n : (0,1) \to \mathbb{R}$$

$$\lim_{n \to \infty} n x^n \qquad 0 < x < 1$$

Write as $\dfrac{x^n}{1/n}$

Ratio Test for sequences:

$$\left| \frac{(n+1) x^{n+1}}{n x^n} \right| = \frac{n+\cancel{1}}{n} |x| \to |x| < 1$$

$$1$$

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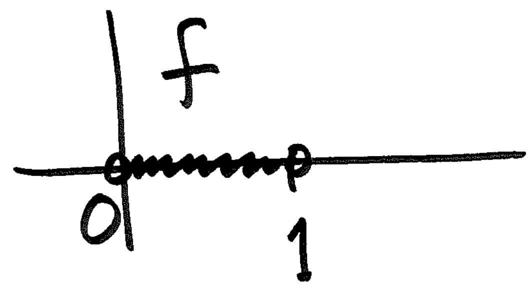
The sequence converges pointwise to the function

$f \equiv 0$



$$\int_0^1 f_n = \int_0^1 n x^n \, dx = n \frac{x^{n+1}}{n+1} \Big|_0^1 = \frac{n}{n+1}$$

$$\lim_{n \to \infty} \int_0^1 f_n = \lim_{n \to \infty} \frac{n}{n+1} = 1$$

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$$\int_0^1 f = \int_0^1 0 = 0$$

$$\int_0^1 f = \int_0^1 \lim f_n = 0 \neq \lim_{n\to\infty} \int_0^1 f_n = 1$$

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Q1: If each f_n is continuous, then is the pointwise limit f also continuous?

Q2. If each f_n is differentiable, then is f differentiable? Further

$$\lim_{n \to \infty} \frac{d f_n}{dx} = \frac{d}{dx} f(x) \,?$$

Q3. If each f_n is integrable, is f also integrable? Is $\lim_{n\to\infty} \int f_n = \int f$?

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Uniform Convergence

Given a sequence $\{f_n\}$, this is said to converge uniformly to f if for any given ε, $\exists N$ such that for all $x \in D$

$$|f_n(x) - f(x)| < \varepsilon, \quad n \geq N$$

N only depends on ε.

Same N works for all x.

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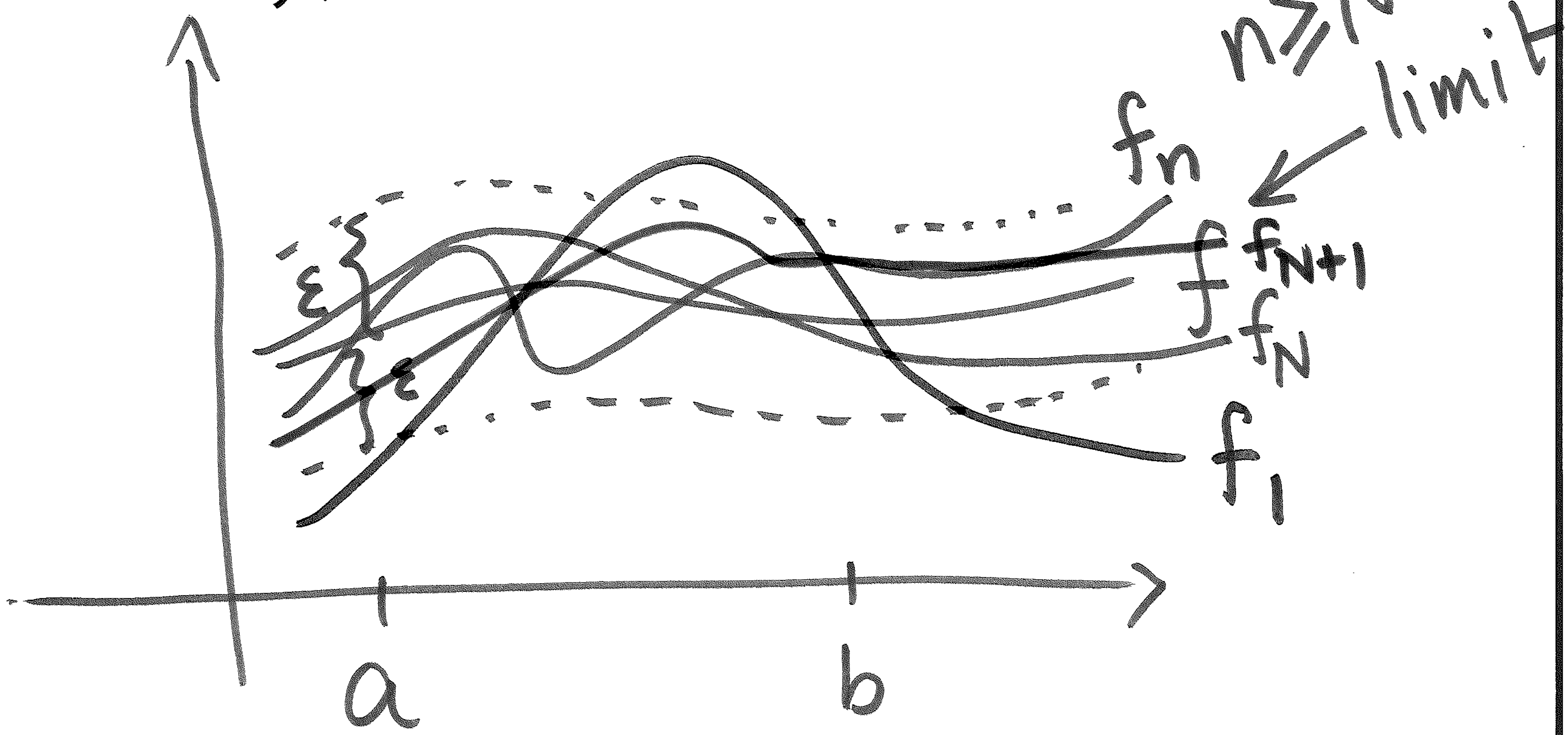

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Example

$f_n : [0,1] \longrightarrow \mathbb{R}$ $\quad f_n(x) = x^n$,

Let $f = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$

We know that $f_n \xrightarrow[\text{pointwise}]{} f$

We will now show that the convergence is not uniform.

Let $\varepsilon = \frac{1}{2}$

Suppose that $\exists\, N$ s.t.

$$|f_n(x) - f(x)| < \frac{1}{2}, \quad n \geq N \ \forall x \in [0,1]$$

$$x_1 = \left(\frac{3}{4}\right)^{\frac{1}{N+1}} \qquad f_{N+1}(x_1) = \frac{3}{4}$$

$$f(x_1) = f\left(\left(\frac{3}{4}\right)^{1/N+1}\right) = 0$$

$$|f_{N+1}(x_1) - f(x_1)| = \left|\frac{3}{4} - 0\right| = \frac{3}{4} > \frac{1}{2}$$

$\Rightarrow \{f_n\}$ is NOT uniformly convergent.