MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 9

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Consider $\{f_n(x) = x^n\}_{n=1}^{\infty}$

$x \in [0, k)$; $k < 1$. Now

$\{f_n\}$ converges uniformly

to $f \equiv 0$

$|f_n(x) - f(x)| = x^n < k^n$

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Given $\varepsilon > 0$, Choose $\underline{N}$ Such that

N ONLY depends on ε $\Leftarrow$ $k^N < \varepsilon$ $\rightarrow$ $k < 1$

Then $x^n < k^n \leq k^N < \varepsilon$; $n \geq N$.

Thus $\{f_n = x^n\}$ is uniformly convergent on $[0, k)$, $k < 1$.

$\{f_n = x^n\}$ converges

a) pointwise to $f = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$

if $f_n : [0, 1] \to \mathbb{R}$, not uniform

b) uniformly to $f \equiv 0$ if

$f_n : [0, R) \to \mathbb{R}$, $R < 1$.

Uniform convergence $\Longrightarrow$ pointwise convergence

Properties of uniform convergence

Theorem : If $\{f_n\}$, $f_n : D \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly to $f : D \longrightarrow \mathbb{R}$ then the limit f is also Continuous .

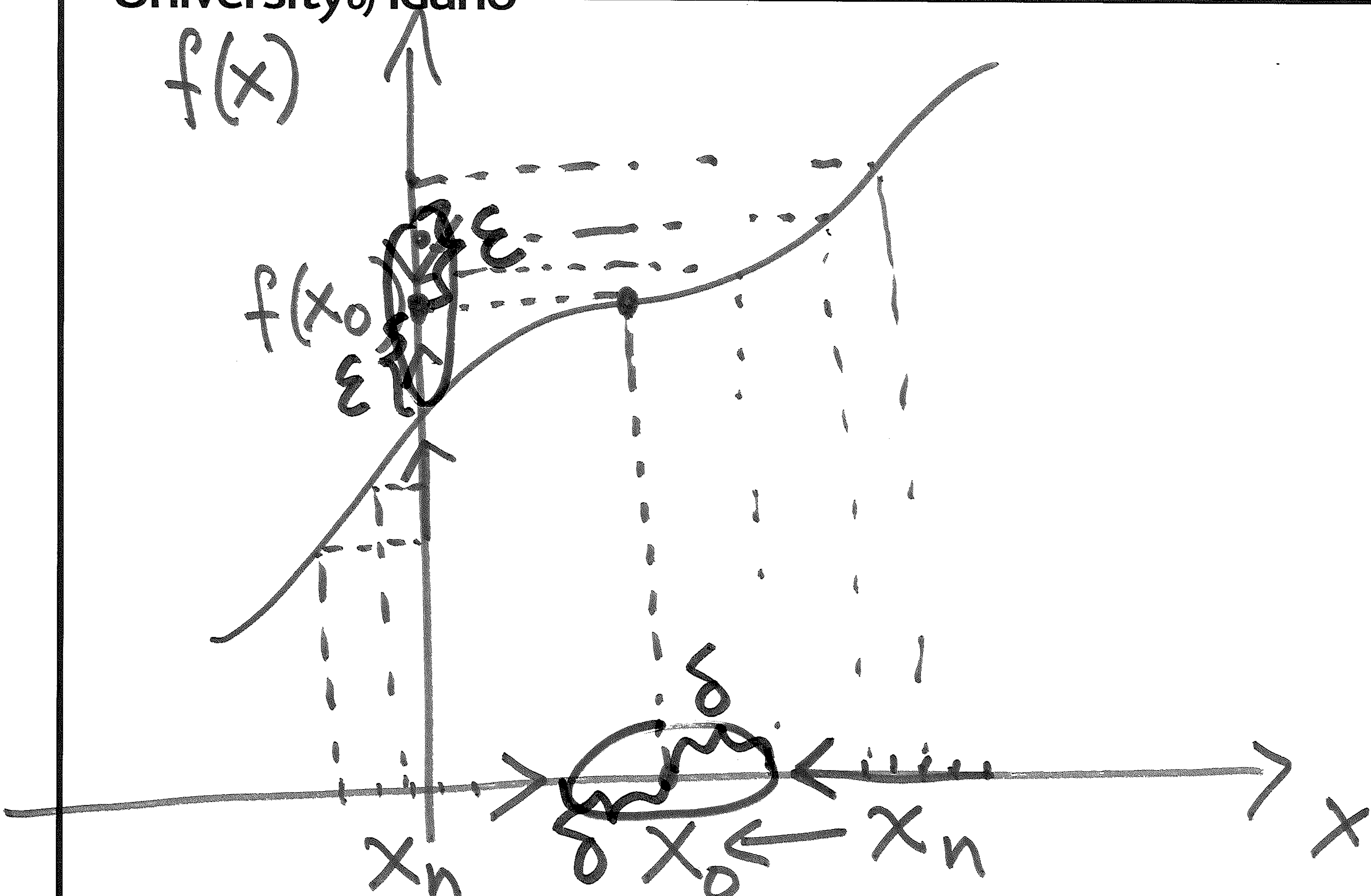
Recall continuity

f is continuous at $x_0 \in D$

if whenever $\{x_n\} \longrightarrow x_0$

then $\{f(x_n)\} \longrightarrow f(x_0)$

the sequence of images


f(x)
ε
f(x₀)
ε
δ
δ
x₀
xₙ
xₙ
x

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Alternatively

f is continuous at x_0 if given $\varepsilon > 0$, $\exists\ \delta$ such that whenever $|x - x_0| < \delta$,

$$|f(x) - f(x_0)| < \varepsilon.$$

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Proof of Theorem

Let $\varepsilon > 0$ be given. By uniform convergence, $\exists N \in \mathbb{N}$ s.t. $\forall x \in D$

(1) $$|f_n(x) - f(x)| < \frac{\varepsilon}{3}, \quad n \geq N$$

Since each f_n is continuous at any point x_0, $\exists \delta_1 > 0$ s.t.

(2) $$|f_N(x_0) - f_N(x)| < \frac{\varepsilon}{3} \text{ whenever } |x_0 - x| < \delta_1, \ x \in D.$$

At the point x_0:

$$
\begin{aligned}
&|f(x) - f(x_0)| \\
&= |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \\
&\le |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}
$$

whenever $|x - x_0| < \delta_1$

TRIANGLE INEQ

$|f(x) - f(x_0)| < \varepsilon$ whenever

$|x - x_0| < \delta_1$

$\Rightarrow$ f is continuous at x_0

Since x_0 was arbitrary, we have continuity of f at every point in D. $\square$