MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 10

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Properties of uniform Convergence (continued)

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From last class:

<u>Thm</u> $\{f_n\} \xrightarrow[\text{unif}]{} f$; f_n is continuous then f is also continuous.

① Ques: <u>Is</u> the converse true? No

If $\{f_n\} \xrightarrow[\text{pointwise}]{} f$, f_n & f are all continuous.

~~If~~ Is the convergence uniform?

Not necessarily

Example: $f_n(x) = \dfrac{1}{nx+1}$, $x \in (0,1)$, $n \in \mathbb{N}$

$f_n \xrightarrow[\text{pointwise}]{} 0 \equiv f$

f_n is continuous for each n

f (the pointwise limit) is also continuous (because it is a constant function)

$f_n \underset{\text{uniformly}}{\not\longrightarrow} f \equiv 0$

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(2) By contrapositiveness, if the pointwise limit f is not continuous then the convergence cannot be uniform.

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Suppose that $f_n \xrightarrow{\text{unif}} f \equiv 0$.

Given ε, $\forall x \in (0,1)$, $\exists N$ s.t.

$$\left| \overset{f_n}{\frac{1}{nx+1}} - \overset{f}{0} \right| < \varepsilon , \quad n \geq N$$

$$\Rightarrow \left| \frac{1}{nx+1} \right| < \varepsilon , \quad n \geq N$$

Let $\varepsilon = \frac{1}{3}$ and let $x = \frac{1}{N}$

At $n = N$: $\left| \frac{1}{N\frac{1}{N}+1} \right| = \frac{1}{2} < \frac{1}{3}$ is $\underline{\underline{\text{not}}}$ true

Thus $f_n \not\xrightarrow{\text{unif}} f \equiv 0$.

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Theorem

Suppose $\{f_n : [a,b] \to \mathbb{R}\}$ is a sequence of continuous functions that converges uniformly to $f : [a,b] \to \mathbb{R}$.
Then f is integrable.
Moreover,

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

[Integration & limit can be interchanged)

 Proof

By the previous theorem, f is continuous. Since continuous functions are integrable, f is also integrable.

Let $L = \int_a^b f$

Need to show: given ε, $\exists N \in \mathbb{N}$ s.t. $\left| \int_a^b f_n - L \right| < \varepsilon$, $n \geq N$

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Since $f_n \xrightarrow{\text{unif}} f$, $\exists\, N_1 \in \mathbb{N}$ s.t. $\forall\, x \in [a,b]$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}, \quad \underline{\underline{n \geqslant N_1}}.$$

$$\left|\int_a^b f_n - L\right| = \left|\int_a^b f_n - \int_a^b f\right|$$

$$= \left|\int_a^b (f_n - f)\right| \leq \int_a^b |f_n - f| < \frac{\varepsilon}{b-a}\int_a^b 1$$

$f_n - f$ is also integrable

when $n \geqslant N_1$

$$= \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$

Therefore, $\left| \int_a^b f_n - L \right| < \varepsilon, \quad n \geq N_1$

Pick $N = N_1$

$\square$

Remark: One can replace the condition of continuity of the f_ns by integrability but the proof is harder