MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 11

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Uniformly convergent sequences of differentiable functions

The uniform limit of a seq of differentiable functions need not be differentiable

$$\left\{ \sqrt{x^2 + \frac{1}{n}} \right\}_{n=1}^{\infty} \xrightarrow{\text{uniform}} |x|$$

(try to show)

$(-1,1)$

f_n — all differentiable

f — not differentiable. (@ $x=0$)

Theorem

I: an open interval

Let $\{f_n\}$ be a sequence of continuously differentiable on I such that

a) $\{f_n\} \xrightarrow[\text{pointwise}]{} f$ on I

b) $\{f_n'\} \xrightarrow[\text{uniformly}]{} g$ on I

Then f is continuously differentiable and $f'(x) = g$ $\underline{\forall x} \in I$.

Def: A function f is said to be continuously differentiable if f is differentiable and its derivative f' is continuous

Proof:

Fix $x_0 \in I$. By the FTOC

Each f_n' is continuous and hence integrable

$$\int_{x_0}^{x} f_n' = f_n(x) - f_n(x_0)$$

$$\lim_{n \to \infty} \int_{x_0}^{x} f_n' = \int_{x_0}^{x} \lim_{n \to \infty} f_n' = \int_{x_0}^{x} g$$

due to Thm in the last lecture

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or, $\lim_{n \to \infty} \left[f_n(x) - f_n(x_0) \right] = \int_{x_0}^{x} g$

due to a)

or $f(x) - f(x_0) = \int_{x_0}^{x} g$

or $f'(x) = g(x)$ due to FTOC

for all $x \in I$

$\square$

Example

$$\{f_n\} = \frac{x^n}{n} \; ; \; 0 \le x \le 1$$

$$f_n \xrightarrow{\text{pointwise}} f \equiv 0 \text{ on } [0,1].$$

$$(f_n \longrightarrow f \text{ uniformly})$$

$$f_n' = x^{n-1} \xrightarrow[\text{NOT uniform}]{\text{pointwise}} g = \begin{cases} 0, & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

$$\left.\begin{array}{l} f_n'(1) = 1 \longrightarrow 1 \\ f'(1) = 0 \end{array}\right\} f' \neq g \text{ @ } x = 1$$

$x \in [0,1]$

Show that $\left\{\frac{x^n}{n}\right\} \longrightarrow 0$ uniformly

Given ε, find N:

$$\left|\frac{x^n}{n} - 0\right| < \varepsilon, \quad n \geq N$$

$$\frac{x^n}{n} < \varepsilon, \quad n \geq N$$

$$\frac{x^n}{n} < \underbrace{\frac{1}{n} < \varepsilon}_{\text{No } x}, \quad n \geq N$$

Pick N satisfying

$$N > \frac{1}{\varepsilon}$$

no dependence on x

Then

$$\frac{x^n}{n} < \varepsilon \quad \forall n \geq N$$

where N does not depend on x