MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 12

Practice Problems

5. $f_n(x) = \dfrac{x^n}{1+x^{2n}}$ $[0,\infty) \longrightarrow \mathbb{R}$

Find the pointwise limit.

uniform?
No

$x=0$: $\{f_n(0) = 0\}_{n=1}^{\infty} \longrightarrow 0$

$x=1$: $\{f_n(1) = \frac{1}{2}\}_{n=1}^{\infty} \longrightarrow \frac{1}{2}$

$f_1(1) = \dfrac{1^1}{1+1^2} = \dfrac{1}{2}$ $\dfrac{1^n}{1+1^{2n}} = \dfrac{1}{2}$

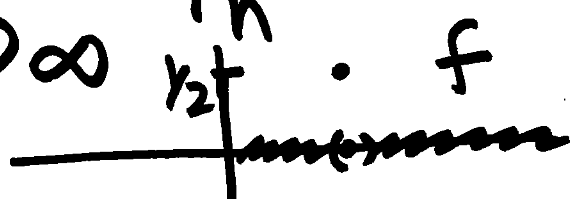
University of Idaho

$0 < x < 1$ $\quad \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^{2n}} = 0$

$1 < x < \infty$ $\quad f_n(x) = \frac{x^n}{1+x^{2n}} = \frac{1}{\frac{1}{x^n}+x^n}$

$$\lim_{n\to\infty} f_n(x) = 0$$

$$\lim_{n\to\infty} f_n(x) = \left.\begin{cases} \frac{1}{2}, & x = 1 \\ 0, & 0 \le x < 1, \\ & 1 < x < \infty \end{cases}\right\} f$$



Each $f_n = \dfrac{x^n}{1+x^{2n}}$ is continuous but the limit f is not. By contrapositiveness (from a thm) the convergence cannot be uniform.

Prob 6

4)

University of Idaho

$$f_n = \frac{x^n}{1+x^{2n}} \qquad [2,5] \longrightarrow \mathbb{R}$$

$$\lim_{n\to\infty} \int_2^5 f_n(x)\,dx$$

Show that $\lim\limits_{n\to\infty} f_n = 0$

<u>uniformly</u>

5

N cannot depend on x.

University of Idaho

Given ε, find N s.t. $\forall x \in [2,5]$

$$|f_n - 0| < \varepsilon, \quad n \geq N$$

or,

$$\frac{x^n}{1+x^{2n}} < \varepsilon, \quad n \geq N$$

$$\frac{x^n}{1+x^{2n}} \leq \frac{5^n}{1+2^{2n}} < \varepsilon$$

independent of $\cancel{\ }$ x

Pick N s.t. $\frac{5^N}{1+2^{2N}} < \varepsilon$

Therefore $\dfrac{x^n}{1+x^{2n}} \xrightarrow{\text{uniformly}} 0$ on $[2, 5]$

Thus $\displaystyle\lim_{n\to\infty} \int_2^5 f_n \overset{\text{due to unif. conv.}}{=} \int_2^5 \underbrace{\lim f_n}_{=0}$

$\displaystyle = \int_2^5 0 = 0.$

#7 $f_n(x) = x e^{-nx^2} \quad x \in [-1, 1]$

(a) Show that f_n converges uniformly to a differentiable function.

$$f_n(x) \longrightarrow$$

$$\frac{x}{e^{nx^2}} \longrightarrow 0$$

given ε, find N s.t. $\forall x$

$$\left| \frac{x}{e^{nx^2}} \right| < \varepsilon, \quad n \geq N$$

$$\times \left| \frac{x}{e^{nx^2}} \right| \leq \frac{1}{e^{nx^2}}$$

maximize: $f_n = x e^{-nx^2}$

$$f_n'(x) = e^{-nx^2} + x e^{-nx^2}(-2nx)$$

a

University of Idaho

$$f_n'(x) = e^{-nx^2}(1 - 2nx^2) = 0$$

$$\Longrightarrow \quad 2nx^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2n}}$$

Do a double derivative test: $x = \frac{1}{\sqrt{2n}}$ gives the max. of f_n.

10

University of Idaho

$$\left|\frac{x}{e^{nx^2}}\right| \le \frac{1}{\sqrt{2n}} e^{-n/2n}$$

$$= \frac{1}{\sqrt{2n}} \frac{1}{\sqrt{e}}$$

Pick N s.t. $\frac{1}{\sqrt{2N}} \frac{1}{\sqrt{e}} < \varepsilon$

$$N > \frac{1}{2e\varepsilon^2}. \quad [\text{no } x]$$

$\Rightarrow$ The convergence is uniform lim & deriv. cannot be interchanged.

$$f_n' = e^{-nx^2}(1-2nx^2)$$

@ $x=0$ $\quad f_n'(0) = 1 \longrightarrow 1$

However the limit is $f \equiv 0$

$$f'(0) = 0 \neq f_n'(0) = 1$$