MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 13

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Series of functions

Today: Power Series

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Definition

$\{a_n\}_{n=0}^{\infty}$ is a sequence of nos.

The series of the form

$$a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

is called a power series centered at $x = a$.

$$\sum_{k=0}^{\infty} a_k (x-a)^k$$

Each term in the series is of the form

$$a_k (x-a)^k$$

$(x-a)$ to the power k

– gives the name of "power series"

$$a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

$$= f_0(x) + f_1(x) + f_2(x) + \cdots$$

$$\left.\begin{aligned} f_0(x) &= a_0 \quad \text{- constant} \\ f_1(x) &= a_1(x-a) \\ f_2(x) &= a_2(x-a)^2 \\ &\vdots \end{aligned}\right\} \text{think } \cancel{\text{of}} \text{ as a sequence of functions}$$

Example

A polynomial of degree n in x:

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

A special kind of power series, finitely many terms

Centered at zero.

Convergence of power series; radius of convergence

At $x = a$: the power series reduces to a_0. Thus every power series converges at least at the center,

$$x = a.$$

Example

Geometric series:

$1 + x + x^2 + x^3 + \cdots$ — center $= 0$

Converges for $|x| < 1$

The sum is $\dfrac{1}{1-x}$

The function $f(x) = \dfrac{1}{1-x}$ has a power series representation $1 + x + x^2 + x^3 + \cdots$ when $|x| < 1$.

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① $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ Center = 0

Ratio: $\displaystyle\lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right|$

$\displaystyle = \lim_{n \to \infty} |x| \frac{1}{n+1} = 0 < 1, \ \forall x$

Converges for all values of x

Radius of Convergence $= \infty$

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(2) $\sum_{k=0}^{\infty} (k!)(x+3)^k$ center $= -3$

Obviouly converges for $x = -3$

Ratio: $\lim_{n\to\infty} \left| \frac{(n+1)!\,(x+3)^{n+1}}{n!\,(x+3)^n} \right|$

$= \lim_{n\to\infty} (n+1)|x+3| = \infty > 1$

if $x \neq -3$

~~Only~~ Convergence only for $x=-3$

Radius of convergence $= 0$

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(3) $\sum_{k=0}^{\infty} \frac{(x-1)^k}{2^k(k+1)}$ Center : $x=1$

Convergence at $x=1$

Ratio: $\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{2^{n+1}(n+2)} \frac{2^n(n+1)}{(x-1)^n} \right|$

$= \frac{|x-1|}{2} \lim_{n\to\infty} \frac{n+1}{n+2} = \frac{|x-1|}{2}$

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Want $\frac{|x-1|}{2} < 1 \implies |x-1| < 2$

$\implies -1 < x < 3$

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Power series converges when

$$-1 < x < 3$$

$\underline{x = -1}$? $\underline{x = 3}$? Ratio $= 1$ $\Rightarrow$ inconclusive

@ $x = -1$: $\displaystyle\sum_{k=0}^{\infty} \frac{(-2)^k}{2^k (k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$

$\displaystyle\lim_{k\to\infty} \frac{1}{k+1} = 0$ ... Alternating Series Test

Convergence @ $x = -1$.

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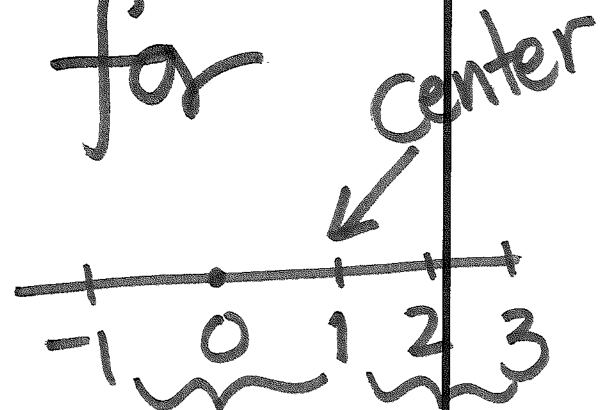
ⓐ $x = 3$: $\displaystyle\sum_{k=0}^{\infty} \frac{\cancel{2^k}}{\cancel{2^k}(k+1)} = \sum_{k=0}^{\infty} \frac{1}{k+1}$

$$= \sum_{k=1}^{\infty} \frac{1}{k}$$

harmonic series which diverges.

Convergence only for

$x \in [-1, 3)$



Radius of convergence = 2

Possibilities in a power series

a) Convergence only the center $(x=a)$

Radius of conv. $=0$

b) Convergence everywhere,
Radius of conv. $=\infty$

c) Convergence on an interval I
s.t. $(a-R, a+R) \subseteq I \subseteq [a-R, a+R]$

Radius of Conv. $=R$