MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 14

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Series of functions (continued)

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Theorem

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

(Radius of conv. $= R$)

(let center be $x=0$)

a) $$\int_0^t f(x)\,dx = \sum_{k=0}^{\infty} a_k \frac{t^{k+1}}{k+1}$$ for $t \in (-R, R)$

b) $$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$ for all $x \in (-R, R)$

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$$\frac{d}{dx} a_0 + \frac{d}{dx} a_1 x + \frac{d}{dx} a_2 x^2 + \cdots$$

$$0 + a_1 + 2a_2 x + \cdots$$

$$= \sum_{k=1}^{\infty} a_k x^{k-1}$$

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Ratio Test

$$f: \lim_{k\to\infty}\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right||x|$$

$$\int f: \lim_{k\to\infty}\left|\frac{a_{k+1}}{k+2}\frac{t^{k+2}}{a_k}\frac{k+1}{t^{k+1}}\right| = \lim_{k\to\infty}|t|\left|\frac{a_{k+1}}{a_k}\right|\underbrace{\cancel{\frac{k+1}{k+2}}}_{1}$$

$$f': \lim_{k\to\infty}\left|\frac{(k+1)a_{k+1}x^{k}}{k\,a_k x^{k-1}}\right| = \lim_{k\to\infty}\underbrace{\cancel{\frac{k+1}{k}}}_{1}\left|\frac{a_{k+1}}{a_k}\right||x|$$

Example

Let $f(x) = \frac{1}{(1-x)^3}$. If $|x| < 1$ find ~~the~~ a power series expansion for $f : (-1, 1) \to \mathbb{R}$.

differentiate

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{k=1}^{\infty} k x^{k-1}$$

$$\frac{2}{(1-x)^3} = 2 + 6x + \cdots = \sum_{k=2}^{\infty} k(k-1) x^{k-2}$$

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$$\Rightarrow \frac{1}{(1-x)^3} = 1 + 3x + \cdots = \sum_{k=2}^{\infty} \frac{k(k-1)}{2} x^{k-2}$$

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Taylor Series

Known

Center is a

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

a_k's are unknown

$$= a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots$$

$$\boxed{f'(a) = a_1}, \quad \boxed{f(a) = a_0}$$

$$f''(x) = 2a_2 + 2\cdot 3\cdot a_3(x-a) + \cdots$$
$$= 2!\,a_2 + 3!\,a_3(x-a) + \cdots$$

$$f''(a) = 2!\,a_2 \implies a_2 = \frac{f''(a)}{2!}$$

$$f'''(x) = 3!\,a_3 + \cdots$$
$$f'''(a) = 3!\,a_3 \implies a_3 = \frac{f'''(a)}{3!}$$

$$\vdots$$

In general, $a_n = \dfrac{f^{(n)}(a)}{n!}$

Thus

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

This power series expansion for f is called the Taylor Series of f about $x = a$.

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If the center $a=0$, then the Taylor series is called a Maclaurin series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Power series solution of diff. equations.

Solve $y'' + y = 0$

Let $y = \sum_{m=0}^{\infty} a_m x^m$ be the solution of the given DE.

$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$, $y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$

The equation becomes

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\left(2a_2 + 3\cdot 2\cdot a_3 x + 4\cdot 3\cdot a_4 x^2 + \cdots\right) + \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) = 0$$

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$$2a_2 + a_0 = 0 \quad : x^0$$

$$3\cdot 2\cdot a_3 + a_1 = 0 \quad : x^1$$

$$4\cdot 3\cdot a_4 + a_2 = 0 \quad : x^2$$

$$\vdots$$

$$a_2 = -\frac{a_0}{2} \quad \Bigg| \quad a_3 = -\frac{a_1}{3\cdot 2} = -\frac{a_1}{3!} \quad \Bigg| \quad a_4 = -\frac{a_2}{4(3)} = \frac{a_0}{4(3)(2)} = \frac{a_0}{4!}$$

$$\vdots$$

a_0, a_1 : arbitrary

$$y = a_0 + a_1 x + \left(-\frac{a_0}{2}\right) x^3 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \cdots$$

a_0, a_1 arbitrary

$\rightarrow$ solves $\quad y'' + y = 0.$

So far in series of funcs. — power series

Next – general series

$$\sum_{k=1}^{\infty} f_k$$

(f_k is any function)