MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 15

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Convergence of series of functions (beyond power series)

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$\{f_n : D \longrightarrow \mathbb{R}\}_{n=1}^{\infty}$ is a sequence of funcs.

$$F_n(x) = \sum_{k=1}^{n} f_k(x)$$: the nth partial sum of $\sum_{k=1}^{\infty} f_k(x)$

We get a new sequence of functions $\{F_n\}_{n=1}^{\infty}$

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The series $\sum_{k=1}^{\infty} f_k$ is said to converge pointwise (uniformly) to $F : D \to \mathbb{R}$ if and only if $\{F_n\}_{n=1}^{\infty}$ converges pointwise (uniformly) to F.

Example

$$f_n : [0,1] \longrightarrow \mathbb{R}, \quad f_n(x) = \frac{x}{(x+1)^{n-1}}$$

Convergence of $\sum_{n=1}^{\infty} f_n(x)$?

When $x=0$: $f_n(0) = 0$ for all n

$$\sum_{n=1}^{\infty} f_n(0) = \sum 0 = 0$$

Let $x \in (0,1]$

$$\frac{1}{2} \le \frac{1}{x+1} < 1$$

$$\sum_{n=1}^{\infty} f_n = x \sum_{n=1}^{\infty} \frac{1}{(x+1)^{n-1}} = x \sum_{n=0}^{\infty} \frac{1}{(x+1)^n}$$

$$= x \frac{1}{1 - \frac{1}{x+1}} = x+1$$

$\sum_{n=1}^{\infty} f_n$ converges pointwise to

$$F(x) = \begin{cases} 0, & x = 0 \\ 1+x, & x \neq 0 \end{cases}$$

Convergence is not uniform since F is not continuous at $x = 0$.

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The Weierstrass M-Test

[test for uniform convergence]

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions and $\{M_n\}_{n=1}^{\infty}$ be a seq. of positive reals such that

$\forall x \in D \quad |f_n(x)| \leq M_n$

If $\sum_{n=1}^{\infty} M_n$ converges then

$\sum_{n=1}^{\infty} f_n$ converges <u>uniformly</u>.

Example (on the M-test)

$$\sum_n \frac{(\sin nx)^2}{n^2} \; ; \; f_n = \frac{(\sin nx)^2}{n^2}$$

$$|f_n(x)| = \left|\frac{(\sin nx)^2}{n^2}\right| \leq \frac{1}{n^2} \rightarrow M_n$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with

$p = 2 > 1 \Rightarrow \sum \frac{1}{n^2}$ converges

and therefore $\sum \frac{(\sin nx)^2}{n^2}$ is uniformly convergent

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Cauchy Criteria for series:

$\{f_n\}$ is a sequence. $\sum f_k$ converges uniformly if & only if given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall x \in D$, $|f_n(x) + \cdots + f_m(x)| < \varepsilon$,

$$n, m \geq N$$

Proof of the M-test.

$\{f_n\}$, $\{M_n\}$ s.t.

$\forall x \in D$, $|f_n(x)| \leq M_n$. Since

the series $\sum_{n=1}^{\infty} M_n$ converges,

by the Cauchy criteria (for nos.)

given ε, $\exists N$ s.t.

$$M_n + \cdots + M_m < \varepsilon,$$

$$n, m \geq N$$

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$$|f_n(x) + \cdots + f_m(x)|$$ by Triangle

$$\leq |f_n(x)| + |f_{n+1}(x)| + \cdots + |f_m(x)|$$

$$\leq M_n + M_{n+1} + \cdots + M_m$$

$$< \varepsilon, \quad n, m \geqslant N$$

By the Cauchy criteria,

$\sum_{n=1}^{\infty} f_n$ converges uniformly $\square$