MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 16

Interchanging $\sum$ and $\int$

" $\sum$ and $\frac{d}{dx}$

$$\{f_n\} \xrightarrow[\text{new seq.}]{} F_n = \sum_{k=1}^{n} f_k$$

$$\downarrow$$

$$\sum_{n=1}^{\infty} f_n$$

pointwise & uniform convergence is same for both

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Theorem

Suppose that $\sum_{k=1}^{\infty} f_k$ is a series of <u>integrable</u> functions that converges <u>uniformly</u> to F on [a, b]. Then F is integrable,

$$\int_a^b F(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$$

$$\int_a^b \sum_{k=1}^{\infty} f_k$$

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Proof

Since $\sum_{k=1}^{\infty} f_k$ converges uniformly we know that $\{F_n\}$ converges uniformly to F.

$F_n = \sum_{k=1}^{n} f_k$, F_n is integrable.

$\lim_{n \to \infty} \int_a^b F_n = \int_a^b F$ (due to a previous thm on uniform convergence of integrable functions)

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$$\lim_{n\to\infty} \int_a^b \sum_{k=1}^{n} f_k = \int_a^b F$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_a^b f_k = \int_a^b F \quad \text{(by linearity)}$$

$$\sum_{k=1}^{\infty} \int_a^b f_k = \int_a^b F$$

$\square$

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Suppose $F_n = nx^n \xrightarrow[x \in (0,1)]{\text{pointwise}} 0 = F$

NOT uniform

$$\int_0^1 nx^n = \frac{n}{n+1} \longrightarrow 1 \neq \int 0 = 0$$

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Example

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

$x \in (-1, 1)$

- $\sum_{k=0}^{\infty} x^k$ converges pointwise on $(-1, 1)$
- The convergence is NOT uniform on $(-1, 1)$
- The convergence is uniform on $[-a, a]$, $a < 1$

part of HW 3.

Let $t \in [-a, a]$

$$\int_0^t \frac{1}{1-x}\,dx = \int_0^t \left(\sum_{k=0}^{\infty} x^k\right) dx$$

$$= \sum_{k=0}^{\infty} \int_0^t x^k\,dx = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}\Bigg|_0^t$$

$$-\ln(1-x)\Big|_0^t = \sum_{k=0}^{\infty} \frac{t^{k+1}}{k+1}$$

$$-\ln(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots$$

$$\ln\frac{1}{1-t} = t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots$$

$$\begin{cases} t \in [-a, a] \\ a < 1 \end{cases}$$

$$t = 1/2: \quad \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^3(3)} + \cdots = \ln 2$$

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Theorem

Let $\{f_n\}$ be a sequence of continuously differentiable functions.

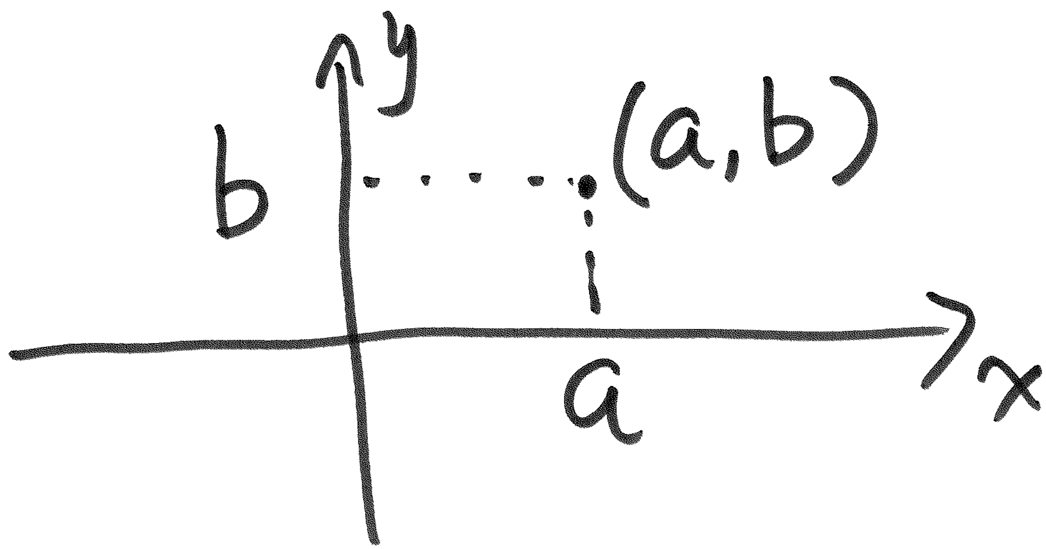
- Let $\sum_{k=1}^{\infty} f_k$ converge <u>pointwise</u> to F
- Let $\sum f_k'$ converge uniformly

Then F is differentiable;

$$\frac{d}{dx}\left(\sum f_k\right) \text{ "} F'(x) = \sum_{k=1}^{\infty} \frac{d}{dx} f_k(x)$$

The Euclidean space $\mathbb{R}^n$.

$\mathbb{R}^2$: (a, b)



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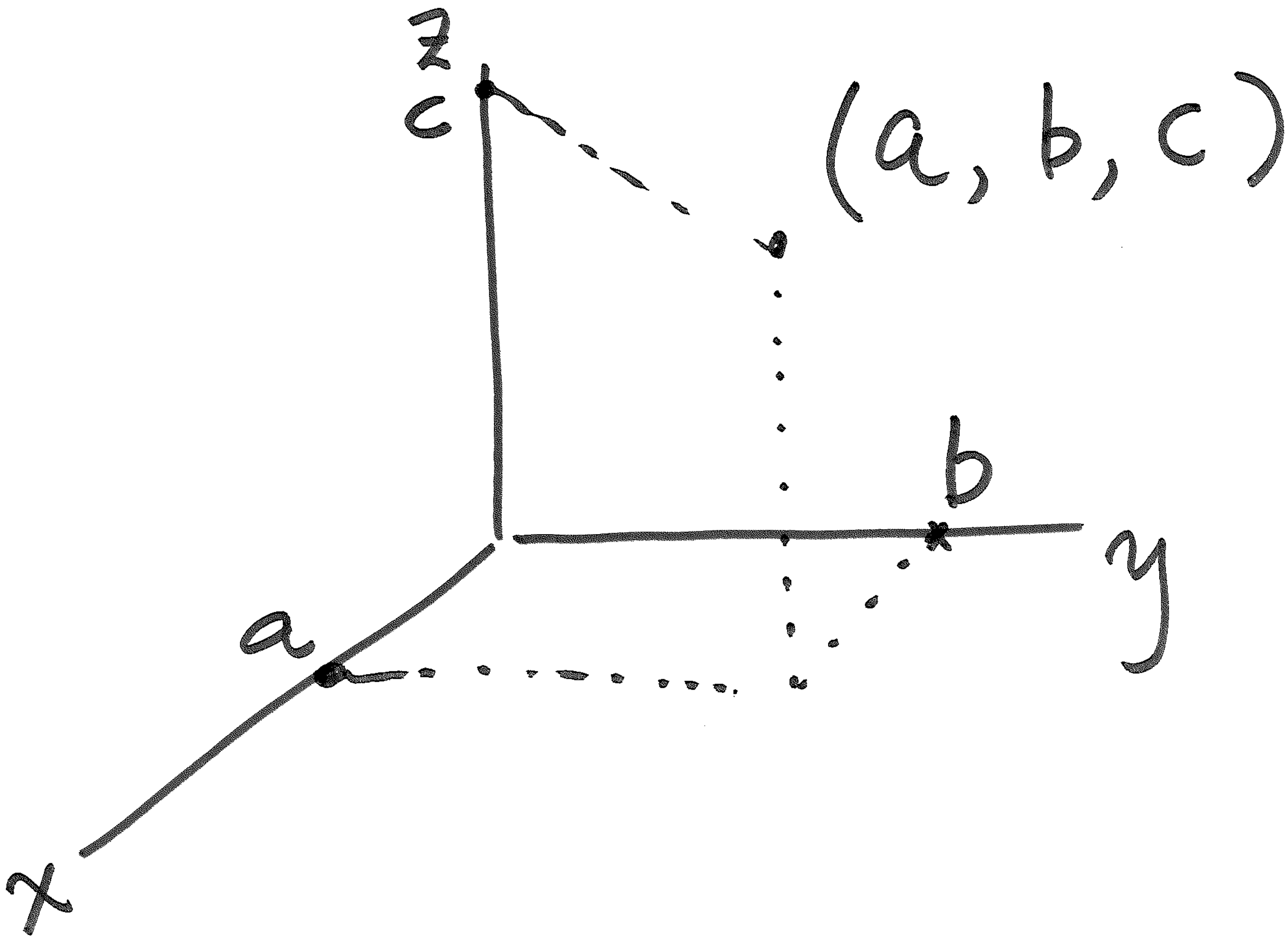
$\mathbb{R}^3$ (a, b, c)



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$\mathbb{R}^n$, $n \in \mathbb{N}$, $\vec{u} \in \mathbb{R}^n$

$\vec{u} = (u_1, u_2, \ldots, u_n)$

each $u_i \in \mathbb{R}$

$\vec{v} = (v_1, v_2, \ldots, v_n)$

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$$\vec{u} = \vec{v} \iff u_i = v_i \,,$$

$$i = 1, 2, \ldots, n$$

$$\vec{u} + \vec{v} = (u_1 + v_1,\ u_2 + v_2,\ \cdots,$$

$$u_n + v_n)$$

$$\alpha \vec{u} = (\alpha u_1, \alpha u_2, \ldots, \alpha u_n)$$

$$\alpha \in \mathbb{R}$$

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Dot product or scalar product

$$\langle \vec{u}, \vec{v} \rangle \quad \text{or} \quad \vec{u} \cdot \vec{v}$$

(inner product)

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v} :=$$

$$u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Properties: (a) Symmetry:

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

(b)

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$\vec{w} = (w_1, w_2, \ldots, w_n)$

(b) Linearity : $\alpha, \beta \in \mathbb{R}$

$$\langle \alpha \vec{u} + \beta \vec{w}, \vec{v} \rangle$$

$$= \alpha \langle \vec{u}, \vec{v} \rangle + \beta \langle \vec{w}, \vec{v} \rangle$$