MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 17

University of Idaho

$\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$

$\vec{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$

$\vec{u} \cdot \vec{v} = \langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

Distance between two points in $\mathbb{R}^n$:

$\vec{u} - \vec{v} = (u_1 - v_1, u_2 - v_2, \dots, u_n - v_n)$

distance $= d(\vec{u}, \vec{v})$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

$\vec{0} = (0, 0, \ldots, 0)$

Norm or length of a point $\vec{u}$ (distance between $\vec{u}$ and $\vec{0}$)

Norm of $\vec{u} = \|\vec{u}\|$

$$= \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

$$= \sqrt{\langle \vec{u}, \vec{u} \rangle}$$

$$d(\vec{u}, \vec{v}) = \sqrt{\langle \vec{u}-\vec{v}, \vec{u}-\vec{v} \rangle} = \|\vec{u}-\vec{v}\|$$

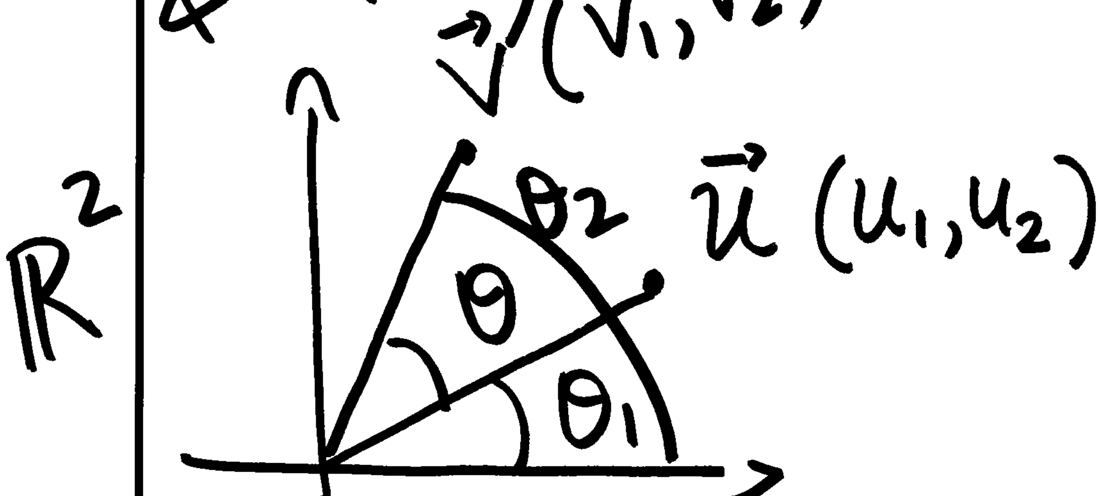
Note: $\|\vec{u}\| \geq 0$, for any $u \in \mathbb{R}^n$

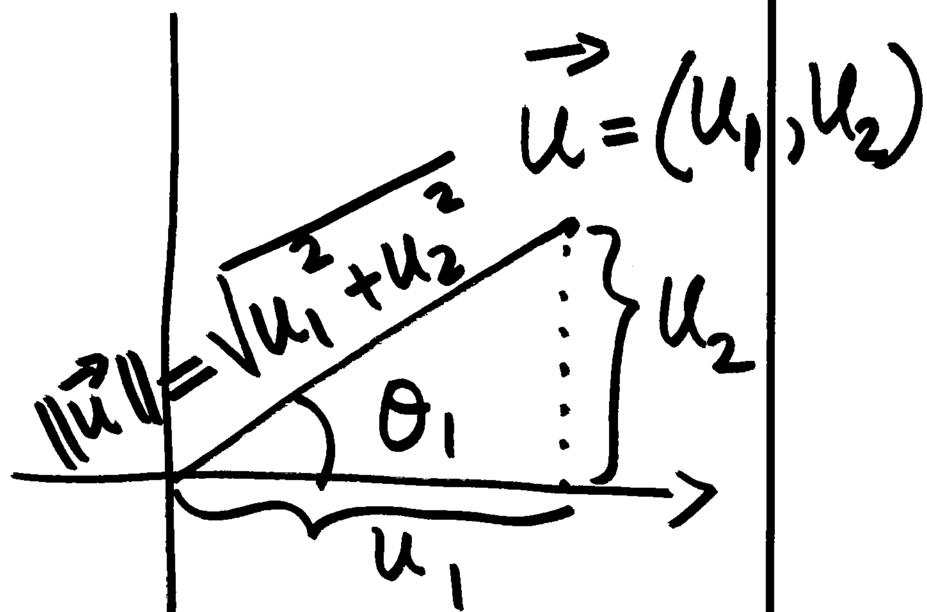
Probosition

Let $\vec{u}$ & $\vec{v} \in \mathbb{R}^2$. Then

$$\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \, \|\vec{v}\| \cos \theta$$

(θ is the angle between $\vec{u}$ & $\vec{v}$)





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$\vec{v}$ makes an angle θ_2 with x-axis

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$u_1 = \|\vec{u}\| \cos\theta_1$, $u_2 = \|\vec{u}\| \sin\theta_1$

$\vec{v} = (v_1, v_2)$, $v_1 = \|\vec{v}\| \cos\theta_2$,

$v_2 = \|\vec{v}\| \sin\theta_2$.

$$\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2$$

$$= \|\vec{u}\| \|\vec{v}\| \cos\theta_1 \cos\theta_2 + \|\vec{u}\| \|\vec{v}\| \sin\theta_1 \sin\theta_2$$

$$= \|\vec{u}\| \|\vec{v}\| \left(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2\right)$$

$$= \|\vec{u}\| \|\vec{v}\| \cos(\theta_1 - \theta_2)$$

$$\theta_2 - \theta_1$$

$$= \|\vec{u}\| \|\vec{v}\| \cos\theta$$

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Notice that $\langle \vec{u}, \vec{v} \rangle = 0$ if and only if $\theta = 90^\circ$ (<u>in $\mathbb{R}^2$</u>) i.e. if $\vec{u}$ & $\vec{v}$ are perpendicular (orthogonal) to each other.

Orthogonality: $\vec{u}$ & $\vec{v}$ (in $\mathbb{R}^n$) are said to be orthogonal if

$$\langle \vec{u}, \vec{v} \rangle = 0.$$

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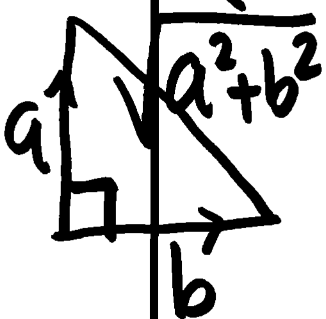
Lemma

$\vec{u}, \vec{v} \in \mathbb{R}^n$ are othogonal

$\Longleftrightarrow \| \vec{u} + \vec{v} \|^2 = \| \vec{u} \|^2 + \| \vec{v} \|^2$

(Pythagorean Identity)



$c^2 = a^2 + b^2$

<u>Proof</u>: $\| \vec{u} + \vec{v} \|^2$

$= \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle$

$= \langle \vec{u}, \vec{u} \rangle + \cancelto{0}{\langle \vec{u}, \vec{v} \rangle} + \cancelto{0}{\langle \vec{v}, \vec{u} \rangle} + \langle \vec{v}, \vec{v} \rangle$

$= \| \vec{u} \|^2 + \| \vec{v} \|^2$ $\square$

Cauchy-Schwarz Inequality

For $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\langle \vec{u}, \vec{v} \rangle| \leq \| \vec{u} \| \| \vec{v} \|$$

<u>Proof</u>: Case 1: If $v = 0$

then $\langle \vec{u}, \vec{0} \rangle = 0$

$$\langle \vec{u}, \vec{0} \rangle = \langle \vec{u}, \vec{v} - \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle - \langle \vec{u}, \vec{v} \rangle = 0$$

$\| \vec{u} \| \| \vec{v} \| = \| \vec{u} \| \cdot 0 = 0$

The result holds trivially.

Case 2: $\vec{v} \neq \vec{0}$, $\lambda = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2}$

$$\begin{aligned}
& \|\vec{u} - \lambda\vec{v}\|^2 \\
& \quad \parallel \\
0 \leq\ & \langle \vec{u} - \lambda\vec{v}, \vec{u} - \lambda\vec{v} \rangle \\
=\ & \langle \vec{u}, \vec{u} \rangle - \lambda\langle \vec{u}, \vec{v} \rangle - \lambda\langle \vec{v}, \vec{u} \rangle + \lambda^2\langle \vec{v}, \vec{v} \rangle \\
=\ & \|\vec{u}\|^2 - 2\lambda\langle \vec{u}, \vec{v} \rangle + \lambda^2\|\vec{v}\|^2
\end{aligned}$$

$$= \|\vec{u}\|^2 - \frac{2\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{v}\|^2} + \frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{v}\|^4} \|\vec{v}\|^2$$

$$= \|\vec{u}\|^2 - \frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{v}\|^2} \geqslant 0$$

$$\|\vec{v}\|^2 \|\vec{u}\|^2 \geqslant \langle \vec{u}, \vec{v} \rangle^2$$

$$\Rightarrow \|\vec{v}\| \, \|\vec{u}\| \geqslant |\langle \vec{u}, \vec{v} \rangle| \qquad \square$$