MATH 472

INTRODUCTION TO ANALYSIS II

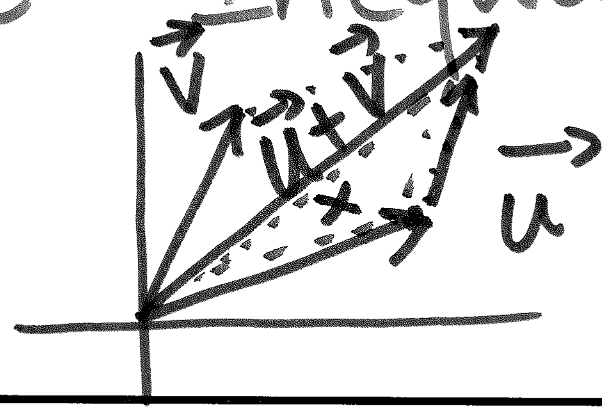
SESSION no. 18

Last Lecture : 1) Pythagorean Identity

2) Cauchy Schwarz

Triangle Inequality

$\mathbb{R}^2$:



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Triangle Inequality

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Proof:

$$\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v} \rangle$$

$$= \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle$$

(by linearity of $\langle \cdot, \cdot \rangle$)

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$$= \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2$$

$$\Big\downarrow \text{use Cauchy-Schwarz}$$

$$\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\,\|\vec{v}\| + \|\vec{v}\|^2$$

$$= (\|\vec{u}\| + \|\vec{v}\|)^2$$

$$\Rightarrow \|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

$\square$

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$$\|\vec{u}-\vec{v}\| \le \|\vec{u}\| + \|-\vec{v}\|$$

$$\| \vec{u} + (-\vec{v}) \|$$

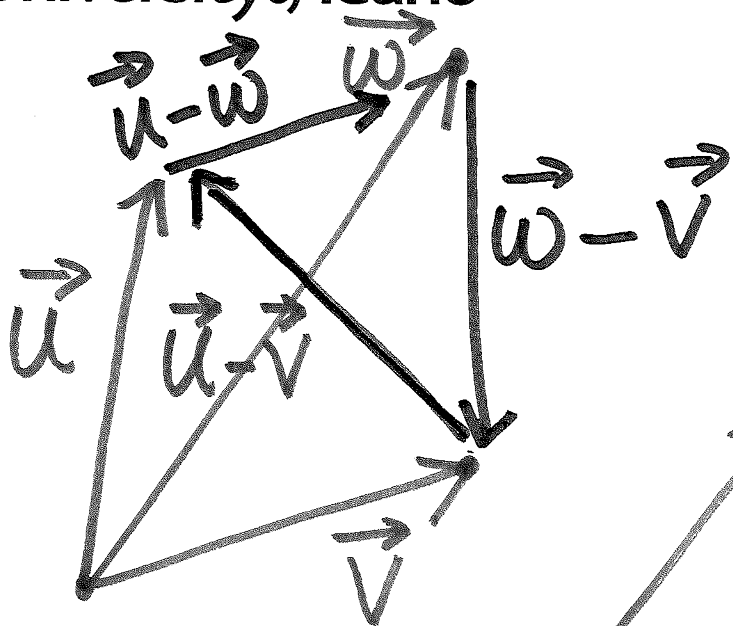
$$= \|\vec{u}\| + \|\vec{v}\|$$

$$\|\vec{u}-\vec{v}\| = \| \underbrace{\vec{u}-\vec{w}} + \underbrace{\vec{w}-\vec{v}} \|$$

$$\le \|\vec{u}-\vec{w}\| + \|\vec{w}-\vec{v}\|$$

$$\Rightarrow d(\vec{u},\vec{v}) \le d(u,w) + d(\vec{w},\vec{v})$$

$d(-,-)$: distance



$$\overbrace{d(\vec{u},\vec{v})}^{\|\vec{u}-\vec{v}\|} \le \underbrace{d(\vec{u},\vec{w})}_{\|\vec{u}-\vec{w}\|}+\underbrace{d(\vec{w},\vec{v})}_{\|\vec{w}-\vec{v}\|}$$

This is an alternate expression for the Triangle Inequality.

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Convergence of sequences in $\mathbb{R}^n$

Let $\{\vec{u}_k\}_{k=1}^{\infty}$ be a seq. in $\underline{\underline{\mathbb{R}^n}}$.

$$\vec{u}_1 = (u_{11}, u_{12}, u_{13}, \ldots, u_{1n})$$

$$\vec{u}_2 = (u_{21}, u_{22}, u_{23}, \ldots, u_{2n})$$

$$\vdots \qquad \downarrow u_1 \qquad \downarrow u_2 \qquad \downarrow u_3 \qquad \downarrow u_n$$

$$\{\vec{u}_k\}_{k=1}^{\infty} \longrightarrow \ ? \ \vec{u} = (u_1, u_2, \ldots, u_n)$$

7 Componentwise convergence criteria

$\{\vec{u}_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ converges to $\vec{u}$ if and only if $\{\vec{u}_k\}$ converges componentwise to $\vec{u}$ i.e.,

if $\vec{u}_k = (u_{k1}, u_{k2}, \cdots, u_{kn})$

and $\vec{u} = (u_1, u_2, \cdots, u_n)$

then $\{\vec{u}_k\} \to \vec{u} \iff \{u_{ki}\} \to u_i$

$1 \le i \le n$

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$x = \frac{1}{k} \quad y = 1/k^2 \qquad x^2 = y$

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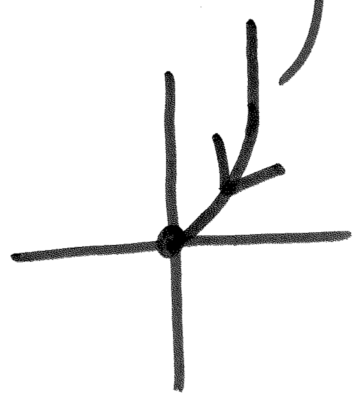
$\{\vec{u}_k\}$ is a seq. in $\mathbb{R}^2$ given by $\quad \vec{u}_k = \left(\frac{1}{k}, \frac{1}{k^2}\right) \in \mathbb{R}^2$

$\vec{u}_1 = (1, 1)$, $\vec{u}_2 = \left(\frac{1}{2}, \frac{1}{4}\right)$, $\vec{u}_3 = \left(\frac{1}{3}, \frac{1}{9}\right)$ ...

$\frac{1}{k} \longrightarrow 0$

$\frac{1}{k^2} \longrightarrow 0$

$y = x^2$



$\searrow$

$\vec{u} = (0, 0)$

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Definition (convergence in $\mathbb{R}^n$)

Let $\{\vec{u}_k\}$ be a sequence in $\mathbb{R}^n$ and let $\vec{u} \in \mathbb{R}^n$. Then $\{\vec{u}_k\}$ is said to converge to $\vec{u}$ if given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$

$$\| \vec{u}_k - \vec{u} \| < \varepsilon, \quad k \geq N$$

or, $d(\vec{u}_k, \vec{u}) < \varepsilon, \quad k \geq N.$

Open sets in $\mathbb{R}^n$.

Definition (Open Ball): Given $\vec{u} \in \mathbb{R}^n$ and $r \in \mathbb{R}$, $r > 0$

$$B_r(\vec{u}) = \{\vec{v} \in \mathbb{R}^n : d(\vec{u},\vec{v}) < r\}$$

$\|u - v\| < r$

is the <u>open ball</u> of radius r about $\vec{u}$.

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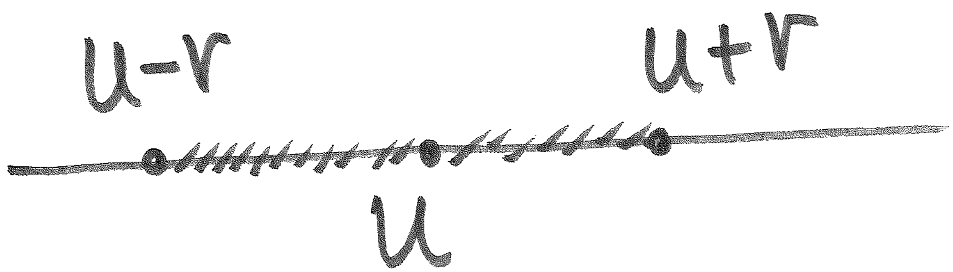
In $\mathbb{R}$:

$d(u,v) = |u - v|$ $\quad B_r(u) =$



$(u-r, u+r)$

In $\mathbb{R}^2$ $\quad \|\vec{u} - \vec{v}\| < r$

$B_r(\vec{u}) =$ circle of radius r, center at $\vec{u}$

Boundary of the circle is NOT included