MATH 472

INTRODUCTION TO ANALYSIS II

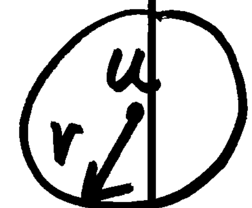
SESSION no. 19

$\|\vec{u}-\vec{v}\|$

$$B_r(\vec{u}) = \{\vec{v} \in \mathbb{R}^n : d(\vec{u},\vec{v}) < r\}$$

open ball about $\vec{u}$, radius r.

Sometimes this is also called a neighborhood of u.



Interior point: $A \subseteq \mathbb{R}^n$. A point $\vec{u}$ is an <u>interior point</u> of A if there is an open ball about $\vec{u}$ contained in A.

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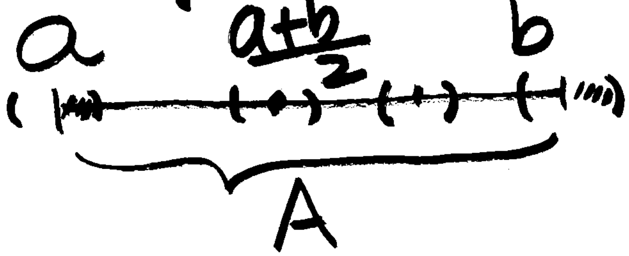
not open

Example $A = (a, b] \subset \mathbb{R}$



a = not an interior pt

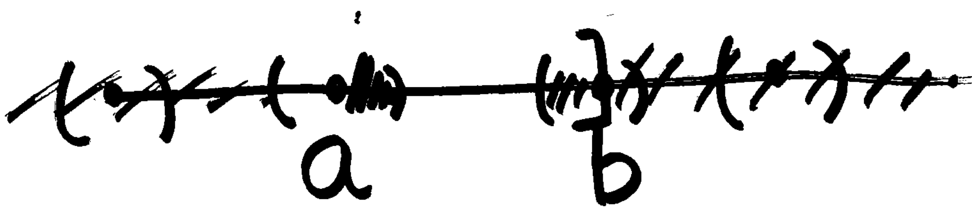
b = not an interior point

$\frac{a+b}{2}$ is an interior point

Any $u \in (a, b)$ is an interior point.

$$\text{int}(A) = (a, b)$$

$A = (a, b]$ $\quad$ $\mathbb{R} \setminus A = (-\infty, a] \cup (b, \infty)$



Is a an interior point of $\mathbb{R} \setminus A$?

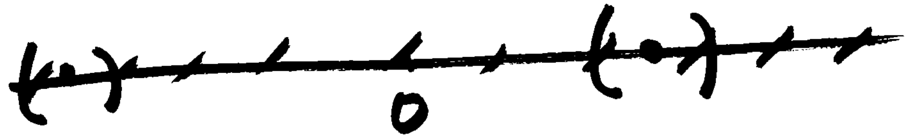
No

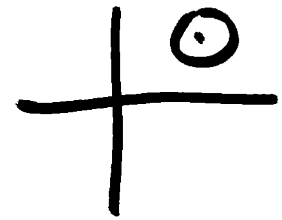
$\text{int}(\mathbb{R} \setminus A) = (-\infty, a) \cup (b, \infty)$

The set of all interior points is called the interior of a set denoted by $\text{int}(A)$

Open sets in $\mathbb{R}^n$

A subset A of $\mathbb{R}^n$ is open if every point in A is an interior point.

1) $\mathbb{R}$  open

2) $\mathbb{R}^n$, $n>1$ also open 

3) empty set ϕ open
[there are no points that contradict the definition]

4) An open ball in $\mathbb{R}^n$ is an open set

Closed Sets in $\mathbb{R}^n$: A set $A \subset \mathbb{R}^n$ is closed if whenever $\{\vec{u}_k\}$ is a sequence in A that converges to $\vec{u}$ then $\vec{u}$ also belongs to A.
Equivalently, A is closed in $\mathbb{R}^n$ if and only if $\mathbb{R}^n \setminus A$ is open.
Roughly, a closed set contains its boundary.

In $\mathbb{R}$: $\quad \overset{A}{(0,1)}$ not closed $\quad \left\{\frac{1}{n+1}\right\}_{n=1}^{\infty} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \longrightarrow 0 \notin (0,1)$

a) $\mathbb{R}^n$ is closed

b) ϕ is closed

because $\mathbb{R}^n \setminus \mathbb{R}^n = \phi$ which is open & $\mathbb{R}^n \setminus \phi = \mathbb{R}^n$ which is open

c) $\{a\}$ is closed

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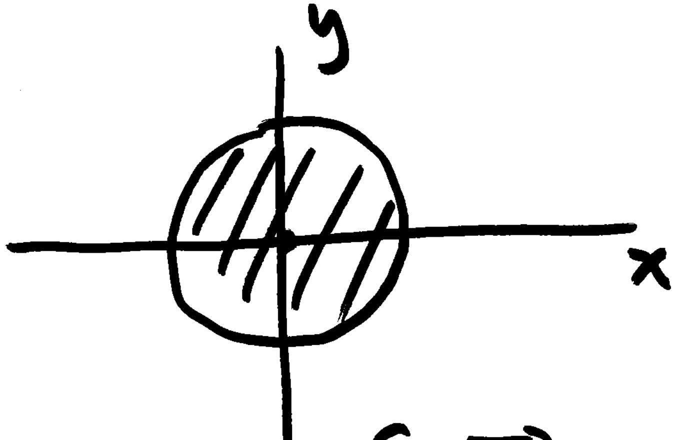
d) $\{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq 1\}$

$\mathbb{R}^2 : \quad \vec{x} = (x, y) \qquad \|\vec{x}\| = \sqrt{x^2+y^2}$

$\|\vec{x}\| \leq 1 \implies x^2+y^2 \leq 1$

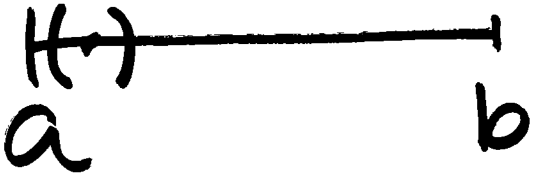


A closed set.

e) $\overline{B_r}(\vec{u}) = \{\vec{v} \in \mathbb{R}^n : d(\vec{v}, \vec{u}) \leq r\}$ is a closed set

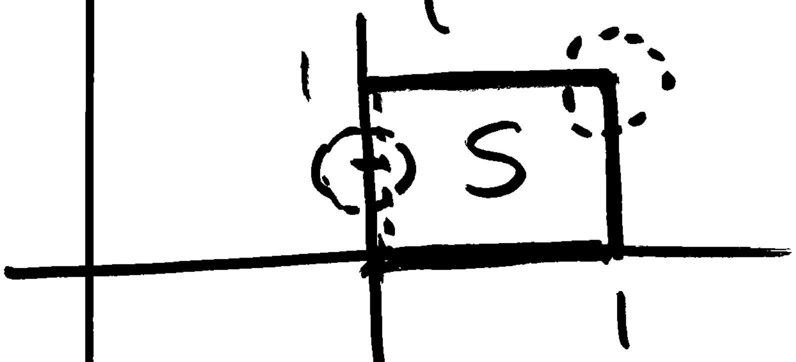
More examples

Any interval (a, b) is an <u>open set</u>.



Every point in (a,b) is an interior point.

$$S = \{(x,y) \in \mathbb{R}^2 : 0 < x \le 1,\ 0 \le y \le 1\}$$



$x=1$, $y=1$ $(1,1)$ is not an interior point but $(1,1) \in S \Rightarrow S$ is NOT open. $\mathbb{R}^2 \setminus S$ is NOT open $(1/2, 0) \in \mathbb{R}^2 \setminus S$ is not an

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interior point of $\mathbb{R}^2 \setminus S$.

$\Rightarrow \mathbb{R}^2 \setminus S$ is not open

$\Rightarrow S$ is not closed