MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 20

1) $\mathbb{R}^n$, ϕ - both open & closed

2) (a, b) - open set in $\mathbb{R}$

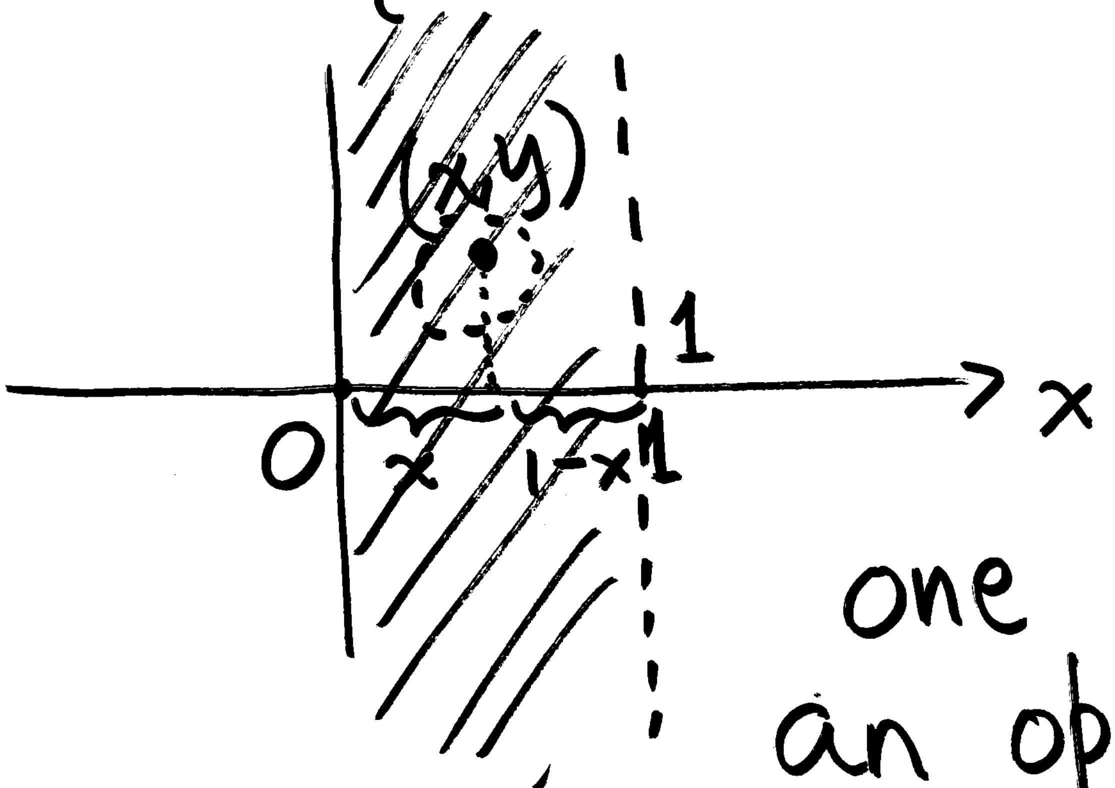
Open interval

3) $[a, b]$ - closed set in $\mathbb{R}$

closed interval

$\{\vec{a}\}$ is closed in $\mathbb{R}^n$

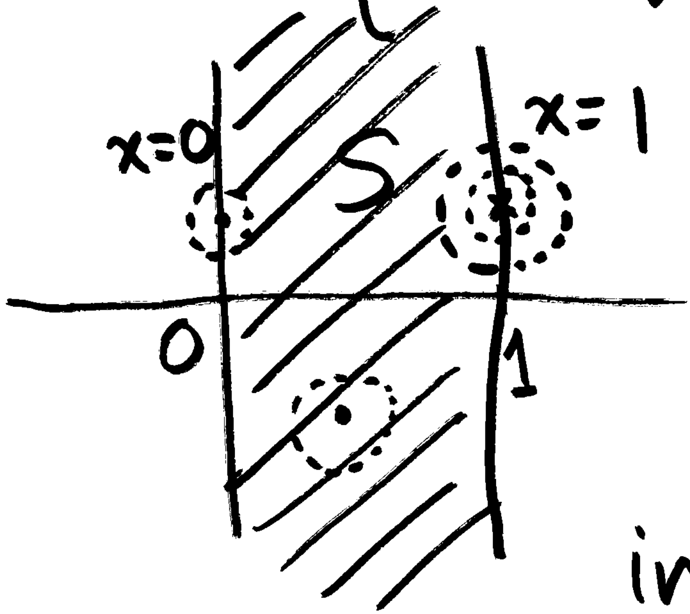
4) $S = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1\}$



Open set

About each $(x,y) \in S$ one can draw an open ball of radius $r < \min(x, 1-x)$ and this is entirely in S.

5) $S = \{(x,y) \in \mathbb{R}^2 : 0 < x \le 1\}$



Open? No. points on the line $x=1$ are not interior points.

Closed? No. $\mathbb{R}^2 \setminus S$ is not open. The line $x=0$ belongs to $\mathbb{R}^2 \setminus S$, but points on $x=0$ are not interior points of $\mathbb{R}^2 \setminus S$.

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Theorem

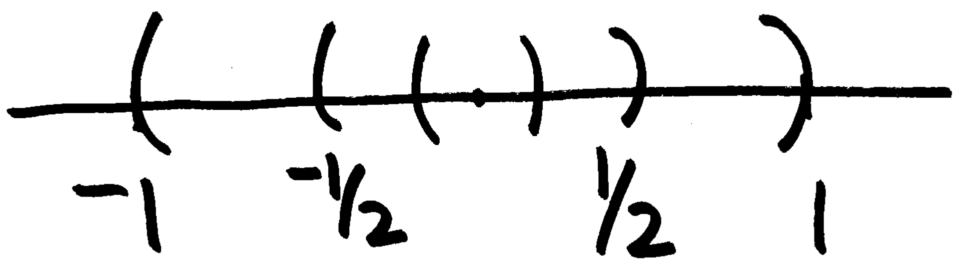
a) The intersection of a finite number of open sets is an open set

b) The union of an arbitrary collection of open sets is open.

$$A_n = \left(-\frac{1}{n}, \frac{1}{n}\right), \quad n = 1, 2, 3, \ldots$$

Take the intersection (not finite)

$$\bigcap_{n=1}^{\infty} A_n = (-1, 1) \cap \left(-\frac{1}{2}, \frac{1}{2}\right) \cap \left(-\frac{1}{3}, \frac{1}{3}\right) \cap \cdots = \{0\}$$



$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

CLOSED.

$\Rightarrow$ Arbitrary intersection of open sets may no longer be open.

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Proof

a) Sufficient to prove for two sets.

Let A & B be open; $C = A \cap B$.

Want to show that C is open.

If $C = \phi$ then C is open.

Let $C \neq \phi$ and take $\vec{x} \in C$.

Since A & B are open, $\exists\ \varepsilon, \varepsilon'$

s.t. $B_\varepsilon(\vec{x}) \subset A$ & $B_{\varepsilon'}(\vec{x}) \subset B$.

Let $r = \min(\varepsilon, \varepsilon')$

University of Idaho Theorem

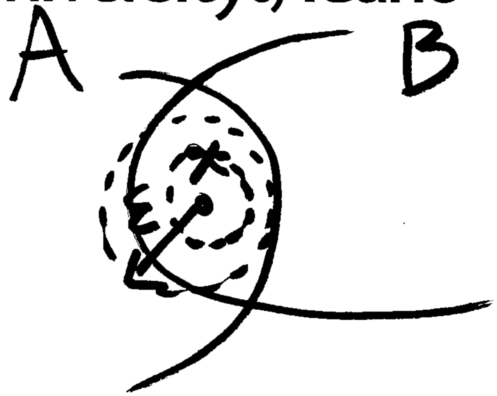
a) The union of a finite number of closed sets is closed

b) The intersection of an arbitrary number of closed sets is closed

Arbitrary no. of closed sets & their ~~intersection~~ union, $A_n = [\frac{1}{n}, 1]$

$n = 1, 2, 3, \ldots$

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = (0, 1] \longrightarrow \text{not closed}$$



Then $B_r(\vec{x}) \subset A$

and $B_r(\vec{x}) \subset B$

$\Rightarrow B_r(\vec{x}) \subset A \cap B = C$

$\Rightarrow \vec{x}$ is an interior point of C

$\Rightarrow C$ is open since $\vec{x}$ was arbitrary

b) Let $U_1, U_2, \ldots$ be open sets

$\bigcup_{i=1}^{\infty} U_i = A$. Want to show that A is open.

Let $\vec{x} \in A$. Then $\vec{x} \in U_k$ for some k. Since U_k is open $\exists$ an open ball $B_\varepsilon(\vec{x}) \subset U_k \subset A$

$\Rightarrow \vec{x}$ is an interior point of A

$\Rightarrow A$ is open

$\square$

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