MATH 472

INTRODUCTION TO ANALYSIS II

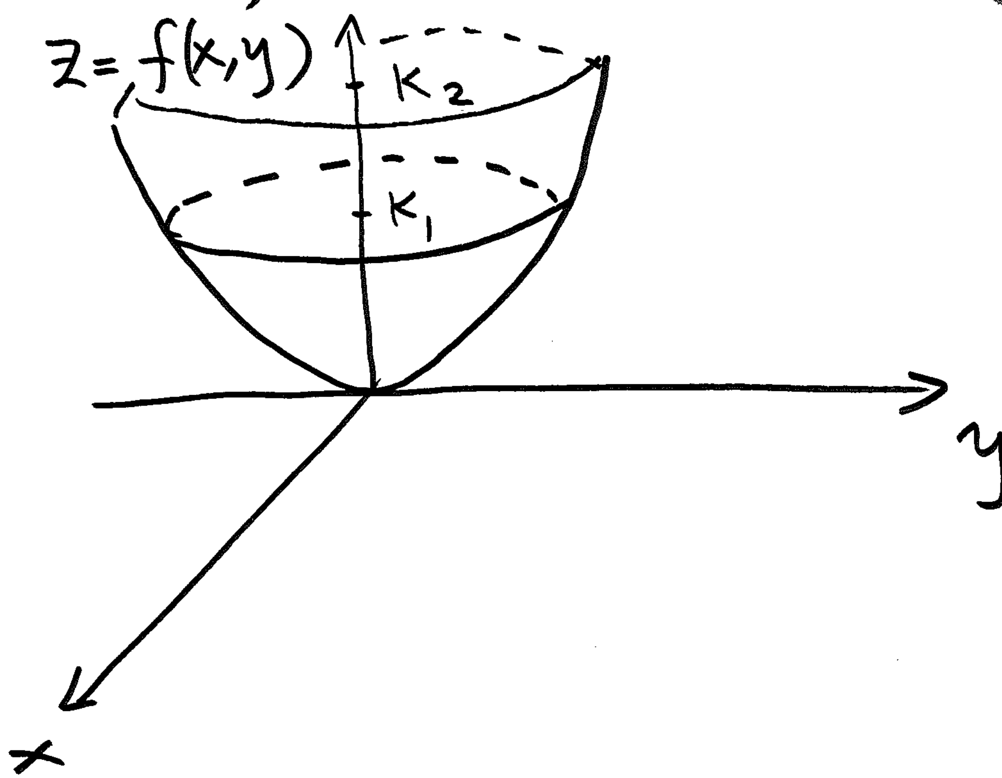
SESSION no. 21

Limits of functions (of several variable)

$f : D \longrightarrow \mathbb{R}$, $D \subset \mathbb{R}^n$, $n \geq 1$

$n = 2$, $D \subseteq \mathbb{R}^2$

$f(x,y) = x^2 + y^2$

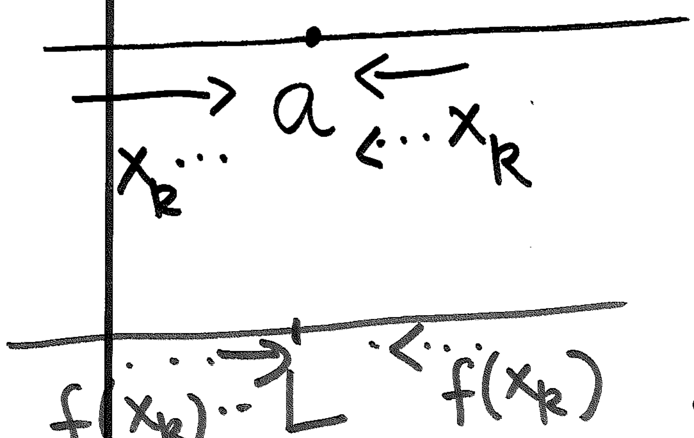


$z = K_1 \Rightarrow x^2 + y^2 = K_1$

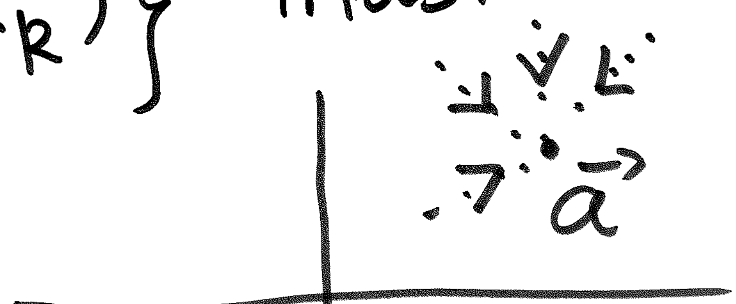
$K_2 > K_1$

$z = K_2 \Rightarrow x^2 + y^2 = K_2$

In $\mathbb{R}$: $\lim\limits_{x \to a} f(x) = L$ means whenever we have a sequence $\{x_k\}$ that converges to a, the sequence $\{f(x_k)\}$ must converge to L.



$\mathbb{R}^2$:



$\vec{a} = (a_1, a_2)$

$|f(x) - L| < \varepsilon$

whenever $|x - a| < \delta$

In $\mathbb{R}^n$: $\vec{x} = (x_1, x_2, \dots, x_n)$

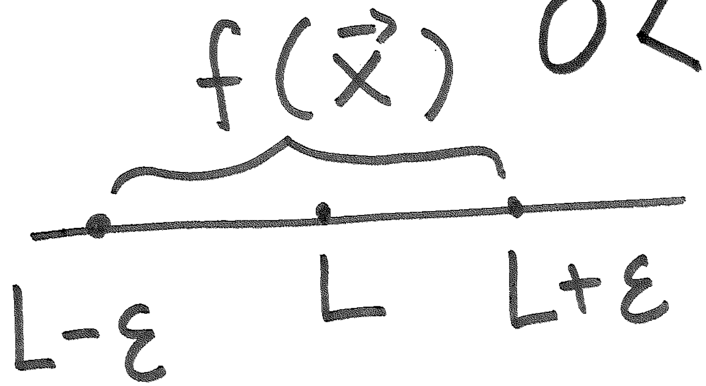
$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L \qquad \vec{a} = (a_1, a_2, \dots, a_n)$$

if whenever a sequence $\{\vec{x}_k\}$ $\cancel{\subseteq \mathbb{R}^n}$ in $\mathbb{R}^n$ converges to $\vec{a}$ then $\{f(\vec{x}_k)\}$ converges to L.

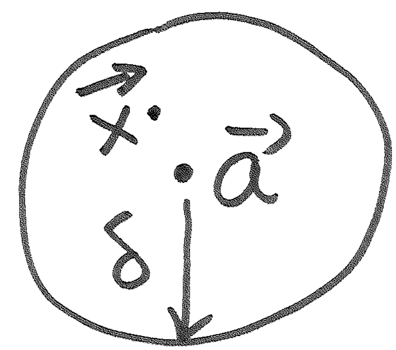
Alternative definition:

$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ if for any ε (given), $\exists$ a δ such that

$|f(\vec{x}) - L| < \varepsilon$, whenever $0 < \|\vec{x} - \vec{a}\| < \delta$.



$\mathbb{R}^2$:



Example

Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

Let $\varepsilon > 0$ be given. Find $\delta > 0$ such that

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon \text{ when } \|(x,y)-(0,0)\| < \delta$$

$$\Rightarrow \left|\frac{3x^2y}{x^2+y^2}\right| < \varepsilon \text{ when } \sqrt{x^2+y^2} < \delta$$

$$\frac{x^2}{x^2+y^2} \le 1$$

$$\frac{3x^2}{x^2+y^2}|y| \le 3|y|$$

$$= 3\sqrt{y^2} \le 3\sqrt{x^2+y^2}$$

$$< \boxed{3\delta < \varepsilon}$$

Pick $\delta < \varepsilon/3$.

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

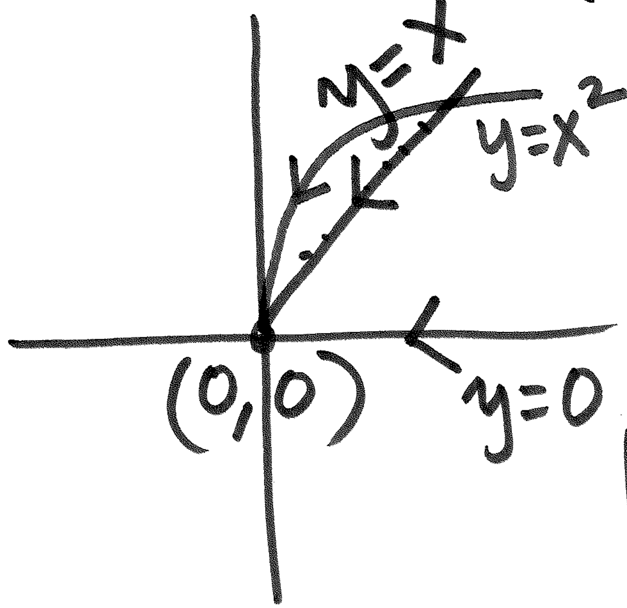
7

Example

University of Idaho

Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2}$ does not exist.

$\mathbb{R}^2$:



Find two paths that approach (0,0) but give different limits.

1) Approach $(0,0)$ along $y=x$:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2x^3}{x^4+x^2} = \lim_{x\to 0} \frac{2x^3}{x^2(x^2+1)}$$

$$= 0$$

2) Approach $(0,0)$ along $y=x^2$:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2x^4}{2x^4} = 1$$

$1 \neq 0$, two limits are different

University of Idaho

Suppose $(x,y) \to (0,0)$ along $y=0$:

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \frac{0}{x^4} = 0$$

Finding the limit:
(showing)

a) If you think the limit exists
guess a value for the limit,
then prove the ε-δ criteria

b) if ~~the~~ you think the limit DNE
then find two <u>different</u>
paths that give different limits.

11

University of Idaho

Continuity:

A function f is continuous at $\vec{x} = \vec{a}$ if

a) $\lim\limits_{\vec{x} \to \vec{a}} f(\vec{x})$ exists (equals some L)

b) $f(\vec{a})$ exists

c) $f(\vec{a}) = L = \lim\limits_{\vec{x} \to \vec{a}} f(x)$.