MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 22

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Continuity

Def: $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$

f is continuous at $\vec{x}_0 \in D$ if given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(\vec{x}) - f(\vec{x}_0)| < \varepsilon$$

whenever

$$\|\vec{x} - \vec{x}_0\| < \delta .$$

$\varepsilon - \delta$ criteria

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Alternate defn. of continuity

$f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$. f is continuous at $\vec{x}_0 \in D$ if whenever $\{\vec{x}_n\}$ converges to $\vec{x}_0$ the sequence $\{f(\vec{x}_n)\}$ converges to $f(\vec{x}_0)$.

Properties of continuity:

Theorem: $f, g : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ are continuous at $\vec{x}_0 \in D$. Then

a) $f \pm g$ is continuous at $\vec{x}_0$

b) fg is " " $\vec{x}_0$

c) f/g " " " $\vec{x}_0$, $g(\vec{x}_0) \neq 0$

d) $|f|$ " " at $\vec{x}_0$

e) $\sqrt{f}$ " " at $\vec{x}_0$

Theorem

$f : A \longrightarrow B$, $g : B \longrightarrow \mathbb{R}$

$f(A) \subseteq B$. If f is continuous at $\vec{x}_0 \in A$, g is continuous at $f(\vec{x}_0) \in B$, then

$\cancel{f \circ g}$ is continuous @ $\vec{x}_0$.

$g \circ f$

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Examples

Let $f(x,y) = x$; $f: \mathbb{R}^2 \to \mathbb{R}$

Let $\vec{x}_0 = (a,b) \in \mathbb{R}^2$.

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} x$$

$$= \lim_{x\to a} x = a = f(a,b)$$

$\Rightarrow f$ is continuous at $\vec{x}_0$.

$\vec{x}_0$ is arbitrary $\Rightarrow f$ is continuous everywhere

$f(x) = x^2 y + e^{xy+1}$

$x^2 y$: product of x^2 & y

x^2 is continuous, y is continuous

$\Rightarrow$ product is continuous

f is the sum, product, composition of continuous and therefore f is continuous.

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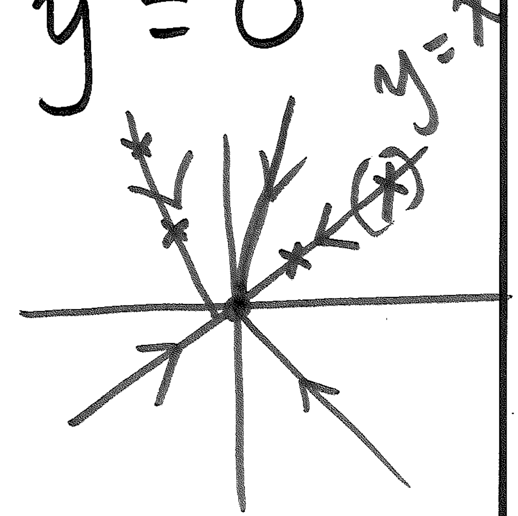
$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & x \neq 0 \text{ or } y \neq 0 \\ 0 & x = y = 0 \end{cases}$$



Along each line $y = mx$ passing through $(0,0)$

$$f(x,y) = \frac{x(mx)}{x^2 + m^2x^2} = \frac{m}{1+m^2}$$ is a constant.

7a)

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$\Rightarrow$ f is continuous at points on each line $y = mx$.

But $\lim_{(x,y)\to(0,0)} f(x,y)$ does NOT exist since a different limit is obtained along each line $y = mx$

$\Rightarrow$ f is NOT cont. at $(0,0)$.

Theorem

Let $f : D \longrightarrow \mathbb{R}$, $D \subset \mathbb{R}^n$. Then the following are equivalent

(i) f is continuous at $\vec{x}_0 \in D$
i.e. ε-δ criteria holds at $\vec{x}_0$

$\Updownarrow$

(ii) Every sequence $\{\vec{x}_k\} \longrightarrow \vec{x}_0$ gives $\{f(\vec{x}_k)\}$ convergent to $f(\vec{x}_0)$.

Show (i) $\Longrightarrow$ (ii)

Let $\{\vec{x}_k\} \longrightarrow \vec{x}_0$. Want to show that $\{f(\vec{x}_k)\} \to f(\vec{x}_0)$.

Let $\varepsilon > 0$. Need to find $N \in \mathbb{N}$ s.t.

$$|f(\vec{x}_k) - f(\vec{x}_0)| < \varepsilon, \quad k \geq N.$$

By (i), $\exists\ \delta > 0$ such that

$$|f(\vec{x}) - f(\vec{x}_0)| < \varepsilon, \quad \|\vec{x} - \vec{x}_0\| < \delta.$$

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Since $\{\vec{x}_k\} \to \vec{x}_0$, for this δ

$\exists\, N_1 \in \mathbb{N}$ s.t.

$$\|\vec{x}_k - \vec{x}_0\| < \delta, \quad k \geq N_1$$

Pick $N = N_1$ then

$\|\vec{x}_k - \vec{x}_0\| < \delta$ and so

$$\|f(\vec{x}_k) - f(\vec{x}_0)\| < \varepsilon, \quad k \geq N_1$$

$\square$