MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 23

Contd. from last lecture

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(ii) $\Longrightarrow$ (i) Suppose that the ε-δ criteria fails.

Given $\varepsilon = \varepsilon_0$, $\nexists$ any δ s.t.

$|f(\vec{x}_0) - f(\vec{x})| < \varepsilon_0$ when $\|\vec{x} - \vec{x}_0\| < \delta$

Let $\delta = \frac{1}{n}$, $n \in \mathbb{N}$, then $\exists\ \vec{x}$ s.t.

(*) $\|\vec{x} - \vec{x}_0\| < \frac{1}{n}$ but $|f(\vec{x}) - f(\vec{x}_0)| \geq \varepsilon_0$

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For a particular n, pick such a $\vec{x}$, call it $\vec{x}_n$. This gives a sequence $\{\vec{x}_n\}_{n=1}^{\infty}$, $\{\vec{x}_n\} \longrightarrow \vec{x}_0$. But by assumption $\{f(\vec{x}_n)\} \longrightarrow f(\vec{x}_0)$. This contradicts (*) and therefore ε-δ condition must hold.

$\square$

Theorem $f: D \to \mathbb{R}$

The following are equivalent:

(i) f is continuous on D

$\Updownarrow$

(ii) For each open set U in $\mathbb{R}$ $f^{-1}(U) \subset D$ is open

Think of $f^{-1}(U)$ as a set S such that $f(S) = U$

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(i) $\Longrightarrow$ (ii)

Let f be continuous and let $U \subseteq \mathbb{R}$, U is open

If $f^{-1}(U) = \varphi$, then we are done. Let $f^{-1}(U) \neq \varphi$.

Take an arbitrary point $\vec{x}_0$ in $f^{-1}(U)$. Show that $\vec{x}_0$ is an interior point.

$f(\vec{x}_0) \in U$ and U is open

Thus $\exists\, r$ s.t. $B_r(f(\vec{x}_0)) \subseteq U$.

Since f is continuous at $\vec{x}_0$,

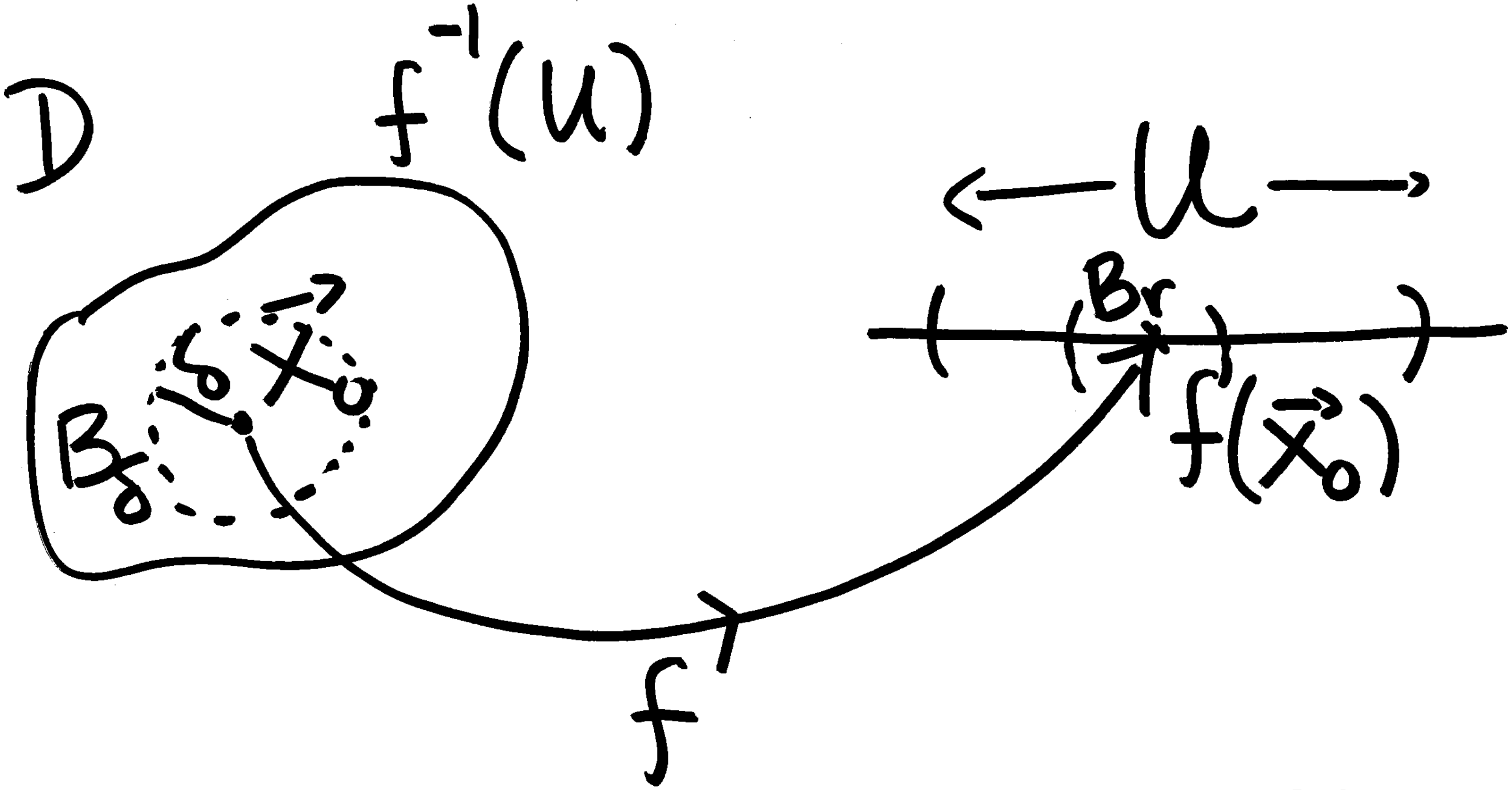
for this r $\exists\, \delta$ s.t.

$$|f(\vec{x}) - f(\vec{x}_0)| < r \text{ when } \|\vec{x} - \vec{x}_0\| < \delta$$

$$\Rightarrow f(x) \in B_r(f(\vec{x}_0)) \subseteq U \text{ when } x \in B_\delta(\vec{x}_0) \subseteq f^{-1}(U)$$

$\Rightarrow$ We have an open ball $B_\delta(\vec{x}_0)$ about $\vec{x}_0$ contained in $f^{-1}(U) \Rightarrow \vec{x}_0$ is an interior point of $f^{-1}(U)$

$\Rightarrow f^{-1}(U)$ is open (since $\vec{x}_0$ was arbitrary) $\square$



$$\vec{x} \in f^{-1}(U) \iff f(\vec{x}) \in U$$

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Example:

$f(\vec{x}) = 1 \qquad \vec{x} \in \mathcal{U}$

Continuous $\qquad \vec{\mathcal{U}}$ is open in $\mathbb{R}^n$

$$f(\vec{\mathcal{U}}) = \{1\}$$

open $\qquad$ closed

Image of an open set can be closed under a continuous function.