MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 24

Partial derivatives

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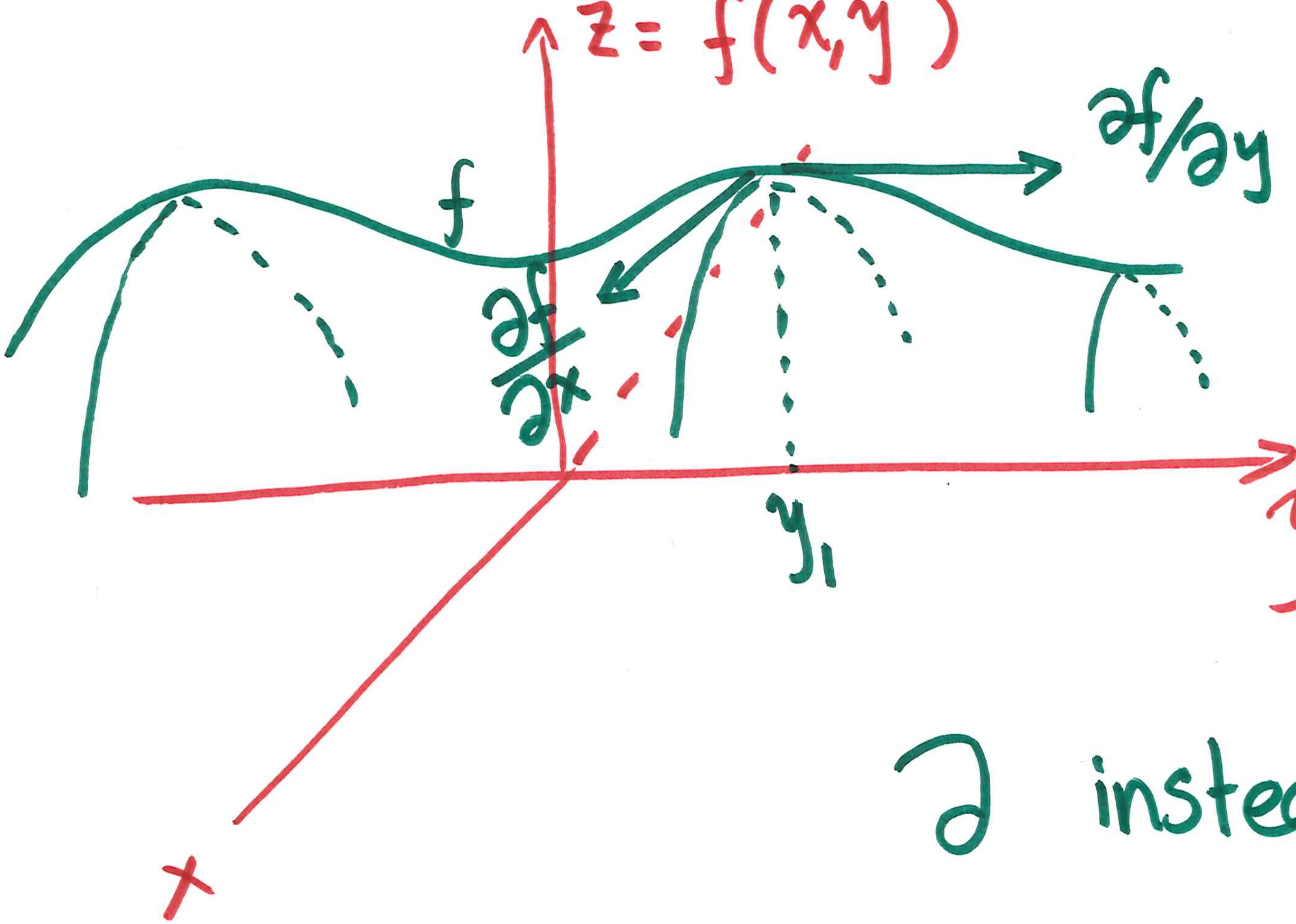
$f : D \longrightarrow \mathbb{R} \qquad D \subseteq \mathbb{R}^2$



∂ instead d

$f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$

D is open

(a) The partial derivative of f w.r.t. x at (a,b) is

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

if this limit exist

(b) w.r.t. y, @ (a,b)

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a,b)}{k}$$

if this exists.

Notation: f_x, $\frac{\partial f}{\partial x}$, f_y, $\frac{\partial f}{\partial y}$

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Example

$$f(x,y) = xy + e^{xy^2}$$

$$f_x = y + y^2 e^{xy^2}$$

$$f_y = x + 2xye^{xy^2}$$

Important note: If f is a function of a single variable then if f is differentiable, it must be continuous.

But this need not be true for funcs. of several variables — even if partial derivatives exist at (a,b) f may not be continuous @ (a,b).

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Example

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

At $(a,b) \neq (0,0)$

$$f_x(a,b) = \frac{\partial}{\partial x}\left(\frac{xy}{x^2+y^2}\right)\Bigg|_{(x,y)=(a,b)} = \frac{y^3 - x^2 y}{(x^2+y^2)^2}\Bigg|_{(x,y)=(a,b)}$$

$$= \frac{b^3 - a^2 b}{(a^2+b^2)^2}$$

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$$f_y(a,b) = \frac{a^3 - ab^2}{(a^2+b^2)^2}$$

At $(0,0)$: $f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$

$$= \lim_{h \to 0} \frac{\frac{h(0)}{h^2+0^2} - 0}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

Therefore, the partial derivatives w.r.t. x & y exist everywhere.

Claim: The function is NOT continuous at (0,0). Consider $\left\{\left(\frac{1}{k}, \frac{1}{k}\right)\right\}_{k=1}^{\infty} \subseteq \mathbb{R}^2$. This seq. converges to (0,0).

$$f\left(\frac{1}{k}, \frac{1}{k}\right) = \frac{\left(\frac{1}{k}\right)^2}{2\left(1/k^2\right)} = \frac{1}{2}$$

Thus $\left\{f\left(\frac{1}{k}, \frac{1}{k}\right)\right\} \longrightarrow \frac{1}{2} \neq 0 = f(0,0)$

Thus f is NOT continuous at (0,0).

If the first order partial derivatives $\partial f/\partial x$ & $\partial f/\partial y$ are <u>Continuous</u> then f is also continuous. This is not the case in the previous example.

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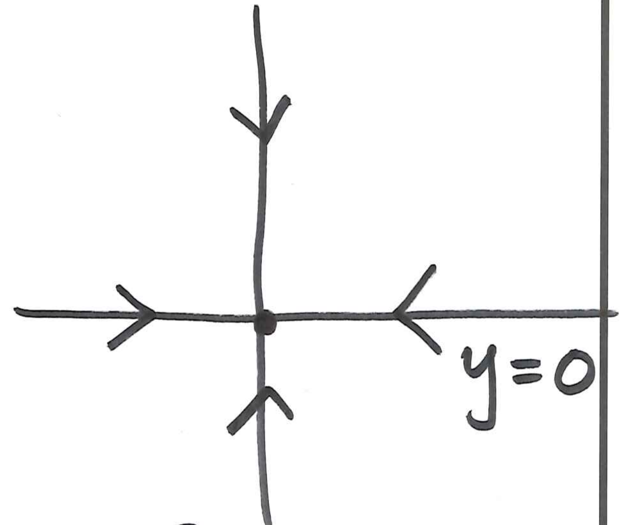
$$\lim_{(x,y)\to(0,0)} f_x = \frac{y^3 - x^2 y}{(x^2+y^2)^2}$$



Along $y=0$: $f_x = 0$

Along $x=0$: $f_x = \dfrac{y^3}{y^4} = \dfrac{1}{y}$

Now $\displaystyle\lim_{(x,y)\to(0,0)} f_x = \lim_{y\to 0} \frac{1}{y} \neq 0$

Thus f_x is not continuous at $(0,0)$.