MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 25

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First order partial derivative

$$\frac{\partial f}{\partial x}, \quad f_x, \quad \frac{\partial f}{\partial y}, \quad f_y$$

If $f = f(x_1, x_2, \cdots, x_n)$

then

$$\frac{\partial f}{\partial x_i}, \quad f_{x_i}$$

Last lecture: an example of f where 1st order partial derivatives exist but f is not continuous

However, if the 1st partial derivatives are continuous then f is continuous.

<u>Continuously</u> <u>differentiable</u> :

f is said to be continuously differentiable if the 1st order partial derivatives exist and are continuous.

Second-order partial derivatives

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ denoted by $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ " " f_{yx} $\frac{\partial^2 f}{\partial y\,\partial x}$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ " " f_{xy} $\frac{\partial^2 f}{\partial x\,\partial y}$

need not be equal

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ " " $\partial^2 f/\partial y^2$ or f_{yy}

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Clairaut's Theorem

If f_{xy} and f_{yx} are continuous then $f_{xy} = f_{yx}$.

Example: $f(x,y) = 3x^2y^3 - \sin x$

$f_x = 6xy^3 - \cos x$, $f_{yx} = 18xy^2$

$f_y = 9x^2y^2$, $f_{xy} = 18xy^2$

continuous

$f_{xy} = f_{yx}$

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Example

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

~~Sh~~ We will show that

$$f_{xy}(0,0) \neq f_{yx}(0,0)$$

f_{xy} & f_{yx} are NOT continuous here.

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$$\frac{\partial f}{\partial y}(x,0) = \lim_{k\to 0} \frac{f(x,k) - f(x,0)}{k}$$

$$= \lim_{k\to 0} \frac{xk(x^2-k^2)}{k(x^2+k^2)} = \lim_{k\to 0} \frac{x(x^2-k^2)}{x^2+k^2}$$

$$= \frac{x^3}{x^2} = x$$

$$\frac{\partial f}{\partial x}(0,y) = \lim_{h \to 0} \frac{f(h,y) - f(0,y)}{h}$$

$$= \lim_{h \to 0} \frac{hy(h^2 - y^2)}{h(h^2+y^2)} = \lim_{h \to 0} \frac{y(h^2-y^2)}{h^2+y^2}$$

$$= -\frac{y^3}{y^2} = -y$$

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$$\frac{\partial^2 f(0,0)}{\partial x \partial y} = \lim_{h \to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h - 0}{h} = 1$$

$$\frac{\partial^2 f(0,0)}{\partial y \partial x} = \lim_{k \to 0} \frac{\frac{\partial f}{\partial x}(0,k) - \frac{\partial f}{\partial x}(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{-k}{k} = -1$$

Thus $\partial^2 f/\partial x \partial y \neq \partial^2 f/\partial y \partial x$

$D \subseteq \mathbb{R}^2$

Directional Derivatives

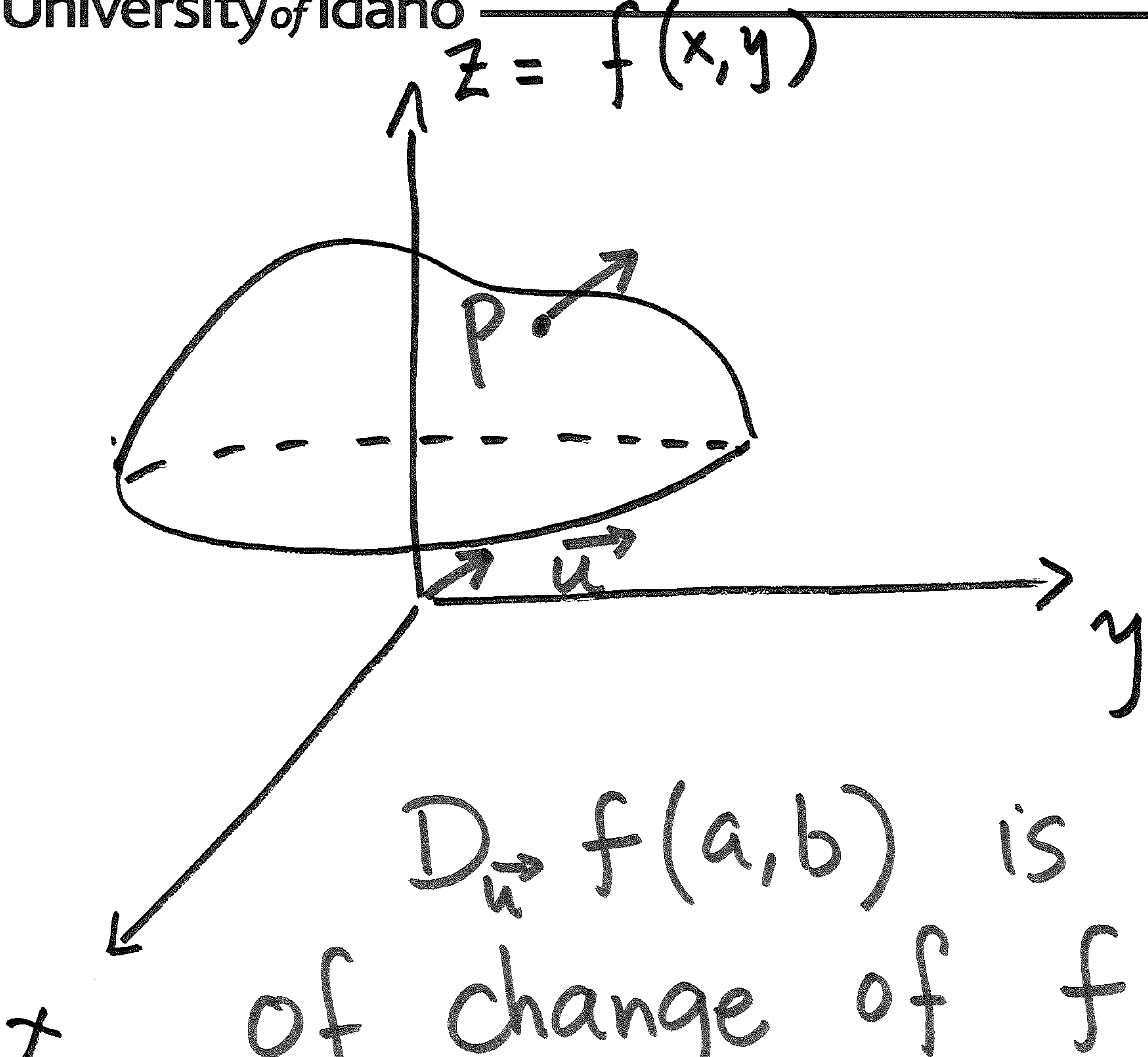
$f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R} \quad P = (a,b) \in D$

$\vec{u} = (u_1, u_2) \in \mathbb{R}^2 , \quad \|\vec{u}\| = 1$

The directional derivative of f at P along $\vec{u}$ is

$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f((a,b) + h(u_1,u_2)) - f(a,b)}{h}$$

$$= \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$



$D_{\vec{u}} f(a,b)$ is the rate of change of f at P along the direction of $\vec{u}$.

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Directional Derivative Thm

Suppose f is continuously differentiable. The directional derivative of f at (a, b) along $\vec{u}$ $(\|u\| = 1)$ is

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u} = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

$$\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right)$$

$$\vec{u} = (u_1, u_2)$$