MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 26

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Directional Derivative

The rate of change along $\vec{u}$ is

$\mathbb{R}^2$

$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

$$\vec{u} = (u_1, u_2), \quad \|\vec{u}\| = 1.$$

$\mathbb{R}^n$

$$D_{\vec{u}} f(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x}+h\vec{u}) - f(\vec{x})}{h}$$

Gradient of $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\nabla f(a,b) = \left(\frac{\partial f(a,b)}{\partial x}, \frac{\partial f(a,b)}{\partial y} \right)$$

Directional Derivative Thm:

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$

$$\|\vec{u}\| = 1$$

projection of the gradient along $\vec{u}$.

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Example

Find the directional derivative of

$$f(x,y) = x^2y^3 - 4y$$

in the direction of $\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ @ $(2,-1)$

$$\|\vec{u}\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$$

$$\frac{\partial f}{\partial x} = 2xy^3 \qquad \frac{\partial f}{\partial y} = 3x^2y^2 - 4$$

4

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$$\frac{\partial f}{\partial x}(2,-1) = -4 \quad , \quad \frac{\partial f}{\partial y}(2,-1) = 12-4=8$$

$$\nabla f(2,-1) = (-4,8)$$

$$D_{\vec{u}} f(2,1) = (-4,8)\cdot\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{4}{\sqrt{2}} + \frac{8}{\sqrt{2}} = \frac{4}{\sqrt{2}}$$

6

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If $\vec{u} = (2,5)$

$$\|\vec{u}\| = \sqrt{2^2+5^2} = \sqrt{29}$$

$$\frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$$ has norm $= 1$.

$$D_{\vec{u}} f(2,-1) = \nabla f(2,-1) \cdot \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$$

$$= (-4,8) \cdot \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right) = \frac{32}{\sqrt{29}}.$$

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Mean Value Theorem

From 471 :

$f : [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b) then $\exists\ c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

7

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Mean Value Lemma

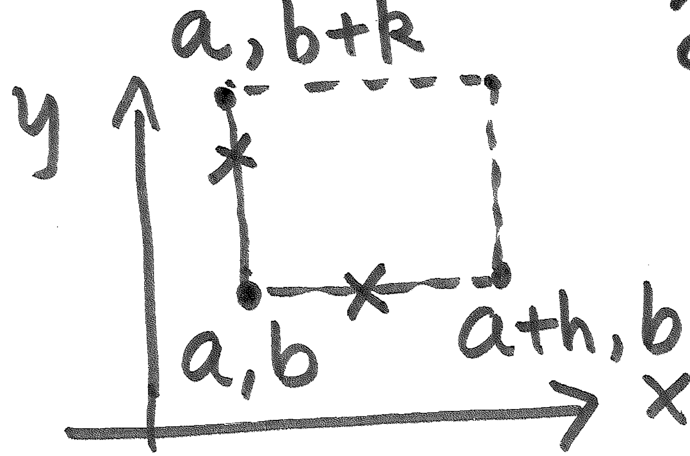
$f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$, D open

$f(x,y)$ has continuous first order partial derivatives.
Then $\exists\ \theta \in (0,1)$ such that for $(a,b) \in D$

$$\frac{f(a+h,b) - f(a,b)}{h} = \frac{\partial f}{\partial x}(a+\theta h, b)$$

and

$$\frac{f(a, b+k) - f(a,b)}{k} = \frac{\partial f}{\partial y}(a, b+\theta k)$$



Mean Value Proposition

$f : B_r(x,y) \longrightarrow \mathbb{R}$, f has continuous first order partial derivatives. If $(x+h, y+k) \in B_r(x,y)$ then $\exists$ $\vec{z}_1$ & $\vec{z}_2 \in B_r(x,y)$ s.t.

$$f(x+h, y+k) - f(x,y) = h\frac{\partial f}{\partial x}(\vec{z}_1) + k\frac{\partial f}{\partial y}(\vec{z}_2)$$

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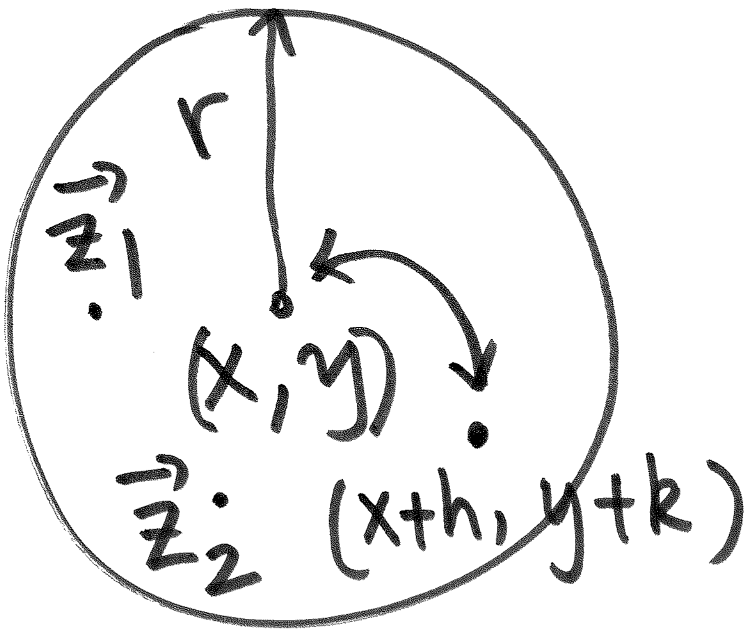

$$\| (x,y) - \vec{z}_i \| \leq \sqrt{h^2 + k^2}$$

$$i = 1, 2,$$

Proof: $f(x+h, y+k) - f(x, y)$

$$= f(x+h, y+k) - f(x+h, y) + f(x+h, y) - f(x, y)$$

$$= k \frac{\partial f}{\partial y}(\underbrace{x+h, y+\theta_1 k}_{\vec{z}_2}) + h \frac{\partial f}{\partial x}(\underbrace{x+\theta_2 h, y}_{\vec{z}_1})$$

By the Mean Value Lemma

$\theta_1, \theta_2 \in (0,1)$

Let $\vec{Z}_2 = (x+h,\ y+\theta_1 k)$

$\vec{Z}_1 = (x+\theta_2 h,\ y)$

Note: $\vec{Z}_1, \vec{Z}_2 \longrightarrow (x,y)$

as $h \to 0,\ k \to 0$

$\square$

Try to use the MV results to prove the directional derivative thm.