MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 27

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Directional Derivative Thm

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$

$\vec{u} = (u_1, u_2)$

$$= \frac{\partial f(a,b)}{\partial x} u_1 + \frac{\partial f(a,b)}{\partial y} u_2$$

where $\|\vec{u}\| = 1$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

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Proof : $\vec{x} \in \mathbb{R}^n$

$$D_{\vec{u}} f(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}$$

$\vec{x} = (a, b) \in \mathbb{R}^2$

$\vec{u} = (u_1, u_2)$

$$f(\vec{x} + h\vec{u}) - f(\vec{x})$$

$$= f(a + hu_1, b + hu_2) - f(a, b)$$

$$= hu_1 \frac{\partial f}{\partial x}(\vec{z}_1) + hu_2 \frac{\partial f}{\partial y}(\vec{z}_2)$$

(by the Mean Value Proposition)

$$\vec{z}_1, \vec{z}_2 \longrightarrow (a,b) \text{ as } h \to 0$$

$$\lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

$$= u_1 \frac{\partial f}{\partial x}(a,b) + u_2 \frac{\partial f}{\partial y}(a,b).$$

$$= \nabla f(a,b) \cdot \vec{u}$$

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In $\mathbb{R}^2$:

$f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$

$D_{\vec{u}} f = \nabla f(a,b) \cdot \vec{u}$

$= \|\nabla f(a,b)\| \, \|\vec{u}\| \cos\theta$

θ = angle between ∇f & u.

max. for $\cos\theta = 1$ i.e. $\theta = 0°$

$\Rightarrow$ The rate of change of f is max along the direction of $\nabla f(a,b)$.

The max rate is thus $\|\nabla f(a,b)\|$ since $\|u\| = 1$.

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Mean Value Theorem

$f: D \to \mathbb{R}$, D open, $D \subseteq \mathbb{R}^n$

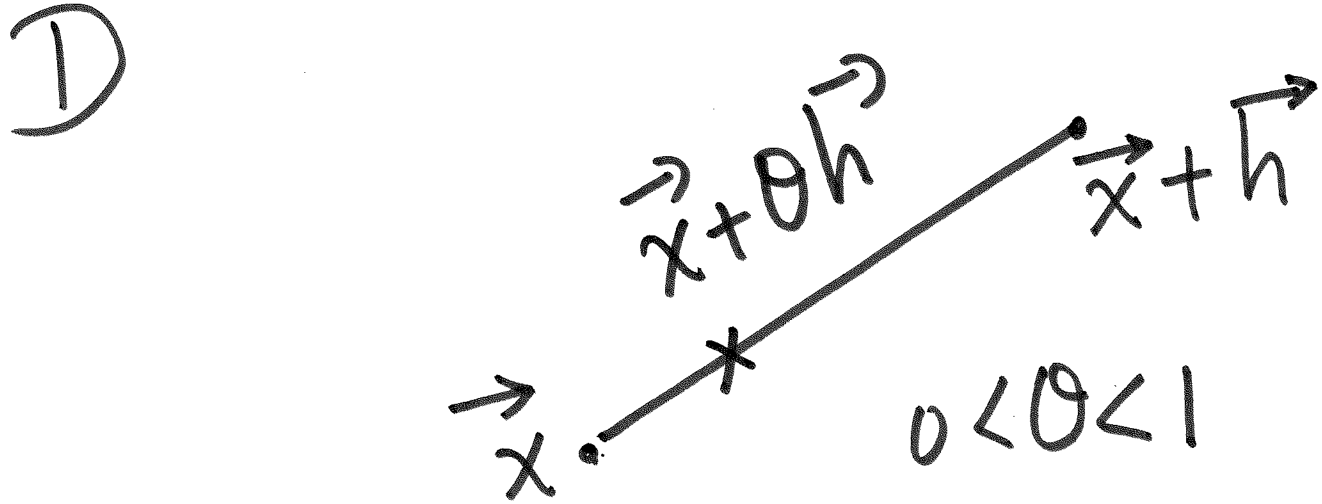
f is continuously differentiable in D.

If the line joining $\vec{x}$ and $\vec{x} + \vec{h}$ lies in D, then $\exists\ \theta$, $0 < \theta < 1$ s.t.

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = \langle \nabla f(\vec{x}+\theta\vec{h}), \vec{h} \rangle$$

$$= h_1 \frac{\partial f}{\partial x_1} + h_2 \frac{\partial f}{\partial x_2} + \cdots + h_n \frac{\partial f}{x_n}$$

6(a)



$\vec{x} = (x_1, \dots, x_n)$ $\quad$ $\vec{h} = (h_1, \dots, h_n)$

$\vec{x} + \vec{h} = (x_1+h_1, \dots, x_n+h_n)$

Theorem

$f : D \xrightarrow{} \mathbb{R}$ ($D \to$ open) is continuously differentiable. Then f is continuous.

[Recall: If the 1st partial derivatives of f exist, the function f need NOT be continuous] Example in a previous lecture

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Proof

Let $\vec{x} \in D$. Since D is open, $\vec{x}$ is an interior point of D. There exist r such that $B_r(\vec{x}) \subseteq D$. Let $\{\vec{x}_k\}$ be a seq. in $B_r(\vec{x})$ that converges to $\vec{x}$. Let $\vec{h}_k = \vec{x}_k - \vec{x}$

By the Mean Value Thm $\exists$ θ_k, $0 < \theta_k < 1$ s.t.

$$f(\vec{x_k}) - f(\vec{x}) = f(\vec{x} + h_k) - f(\vec{x})$$

$$= \left\langle \nabla f(\vec{x} + \theta_k \vec{h_k}), \vec{h_k} \right\rangle$$

Note $\lim_{k \to \infty} \vec{h_k} = \vec{0}$

$$\lim_{k \to \infty} \vec{x} + \theta_k \vec{h_k} = \vec{x}$$

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Since f is continuously differentiable, we further have

$$\lim_{k \to \infty} \nabla f(\vec{x} + \underbrace{\theta_k \vec{h_k}}_{\to 0}) = \nabla f(\vec{x})$$

$$\Rightarrow \lim_{k \to \infty} f(\vec{x_k}) - f(\vec{x}) = 0$$

$$\Rightarrow \lim_{k \to \infty} f(\vec{x_k}) = f(\vec{x})$$

$\Rightarrow f$ is continuous at $\vec{x}$. $\square$