MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 28

Review

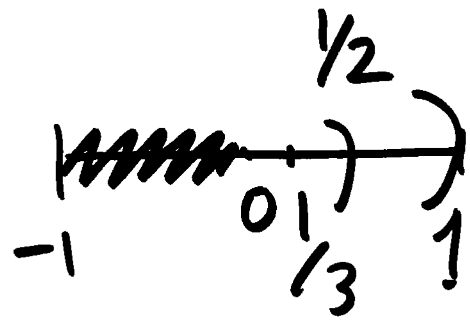
#1

$$\bigcap_{n=1}^{\infty} \left[-1, \frac{1}{n}\right)$$

open? closed? neither? both?

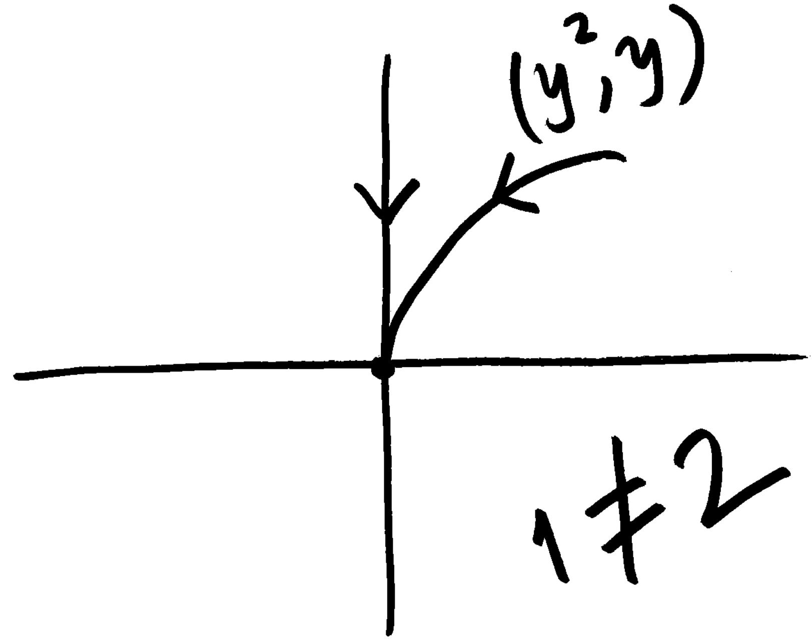
$$[-1, 1) \cap \left[-1, \frac{1}{2}\right) \cap \left[-1, \frac{1}{3}\right)$$

$$\cap \cdots$$



$= [-1, 0]$ is closed.

#4 (a) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{y^2}$ does not exist.



Along $x=0$:

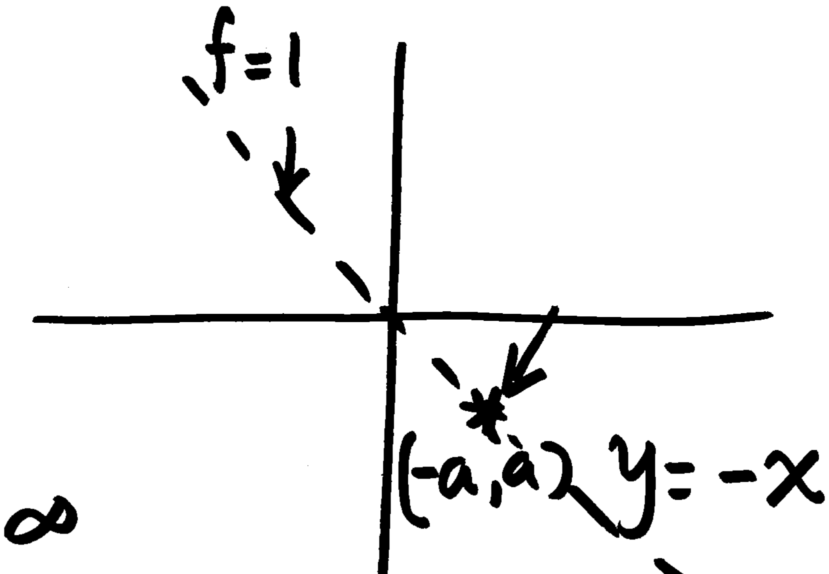
$$\lim_{y\to 0} \frac{y^2}{y^2} = 1$$

$1 \neq 2$

Along $x=y^2$: $\displaystyle\lim_{y\to 0} \frac{y^2+y^2}{y^2} = 2$

#5 Show $f(x,y) = \begin{cases} \dfrac{x-y}{x+y} & y \neq -x \\ 1 & y = -x \end{cases}$

is NOT continuous



Pick a point on

$y = -x$: $(-a,a)$

$\left\{\left(-a+\frac{1}{k},\ a+\frac{1}{k}\right)\right\}_{k=1}^{\infty} \longrightarrow (-a,a)$

$f(-a,a) = 1$

$$f\left(-a+\frac{1}{k},\ a+\frac{1}{k}\right)$$

$$= \frac{-a+\cancel{\frac{1}{k}}-\left(a+\cancel{\frac{1}{k}}\right)}{-\cancel{a}+\frac{1}{k}+\cancel{a}+\frac{1}{k}} = -ak$$

$$\left\{f\left(-a+\frac{1}{k},\ a+\frac{1}{k}\right)\right\} = \left\{-ak\right\}_{k=1}^{\infty} \not\to 1 = f(-a,a)$$

$\Rightarrow$ f is NOT continuous on the line $y=-x$.

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6(b) $f(x,y) = \begin{cases} \dfrac{3x^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Show that f is continuous at $(0,0)$.

Recall

f is cont. at (a,b) if given $\varepsilon > 0$

$\exists\ \delta$ s.t.

$|f(x,y) - f(a,b)| < \varepsilon$ whenever

$\|(x,y)-(a,b)\| < \delta$

Find δ from given ε.

$(a,b) = (0,0)$

Given ε, find δ s.t.

$$\left| \frac{3x^4}{x^2+y^2} \right| < \varepsilon \quad \text{when} \quad \|(x,y)\| < \delta$$

$$\sqrt{x^2+y^2} < \delta$$

$$\frac{x^2}{x^2+y^2} \leq 1$$

$$\frac{3x^4}{x^2+y^2} = 3\frac{x^2}{x^2+y^2}x^2 \leq 3x^2 \leq 3(x^2+y^2)$$

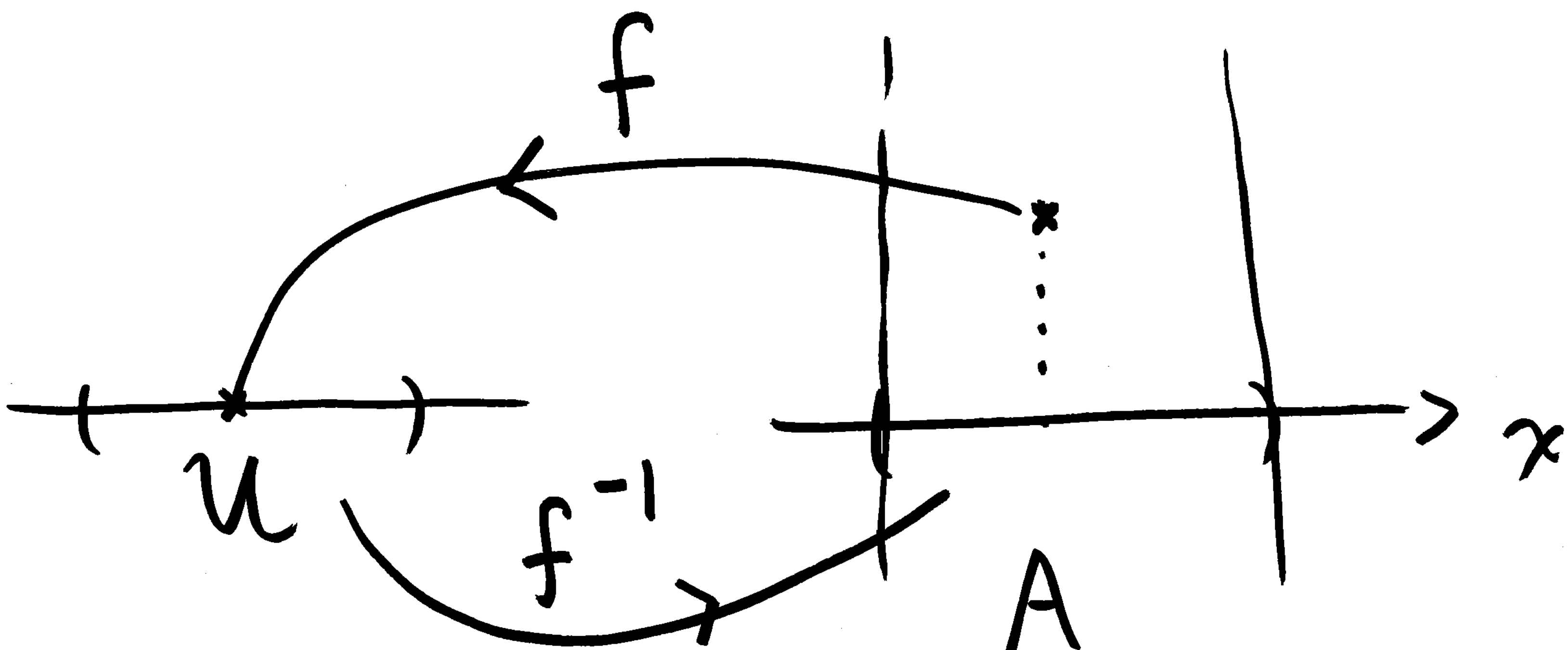
$$= 3\delta^2 < \varepsilon$$

Pick δ s.t. $\delta < \sqrt{\frac{\varepsilon}{3}}$.

#7 $U \subset \mathbb{R}$, U is open

$A = \{(x,y) \in \mathbb{R}^2 : x \in U\}$

Show that A is open.



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Let $f(x,y) = x$

f is continuous.

Think of $f^{-1}(U)$ as the set A such that $f(A) = U$

$A = f^{-1}(U)$, U is open

Therefore, A must be open because the ~~image~~ inverse image of an open set is open under a continuous function.

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#9 $f(x,y) = x + \sin y^2$

$D_{\vec{u}} f(1,0) = ?$ $\quad \vec{u} = (1, -2)$

① Find the gradient:

$$\nabla f(x,y) = \Big(\underbrace{1}_{\frac{\partial f}{\partial x}}, \underbrace{2y \cos y^2}_{\partial f/\partial y}\Big)$$

$$\nabla f(1,0) = (1, 0)$$

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(2) $\|\vec{u}\| = \sqrt{1+4} = \sqrt{5}$ (not of norm 1)

(3) $D_{\vec{u}} f(1,0) = \left\langle \nabla f(1,0), \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right) \right\rangle$

$= \left\langle (1,0), \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right) \right\rangle$

$= \frac{1}{\sqrt{5}} + 0 = \frac{1}{\sqrt{5}}$