MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 29

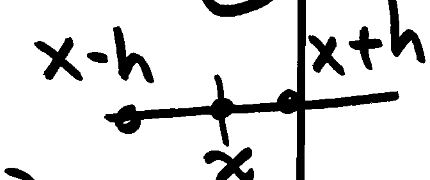
Approximation of functions

[Taylor polynomials]

$f : \underset{\subseteq \mathbb{R}}{D} \longrightarrow \mathbb{R}$ (func. of a single variable)

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - [f(x) + hf'(x)]}{h} = 0$$



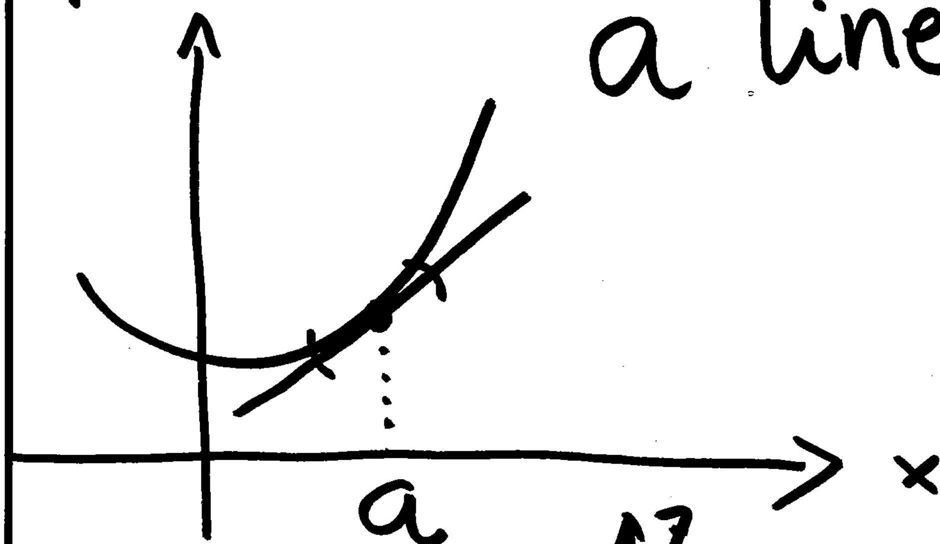
or, $f(x+h) \approx f(x) + hf'(x)$

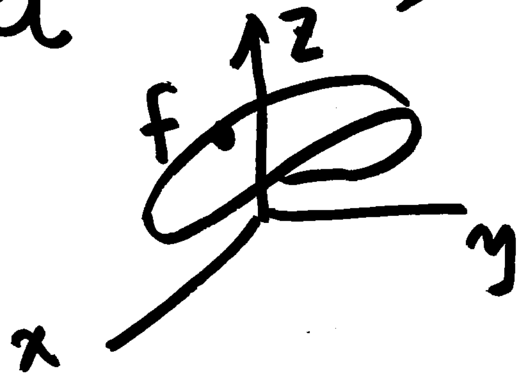
$f(a) - a f'(a) + x f'(a)$

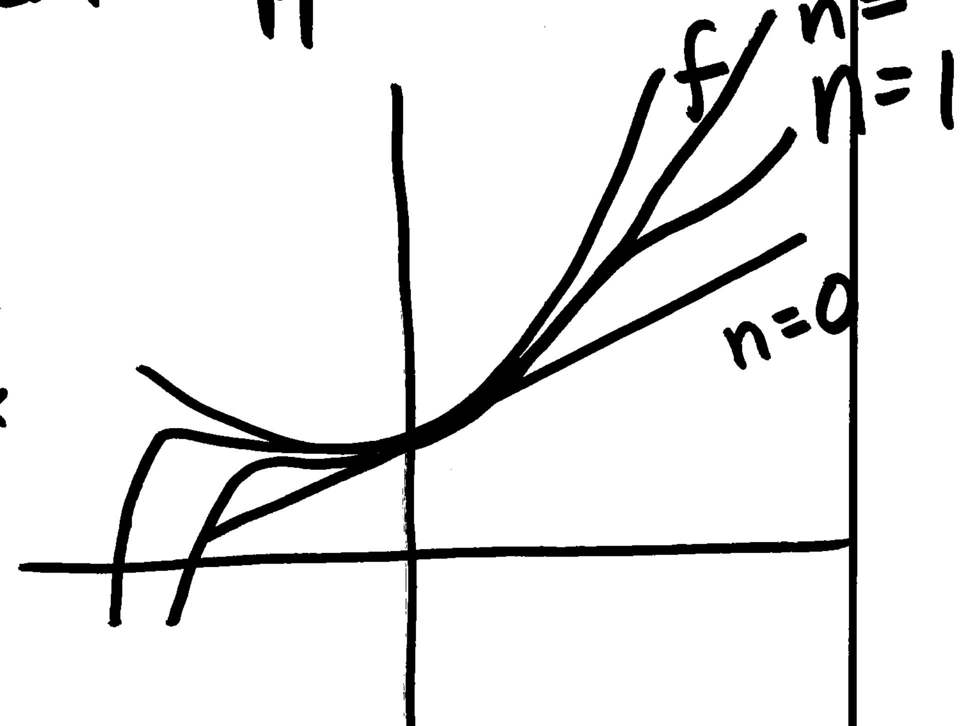
or, $f(x) \approx \underbrace{f(a) + (x-a) f'(a)}$ is an approximation of f

a linear approximation

f(x)





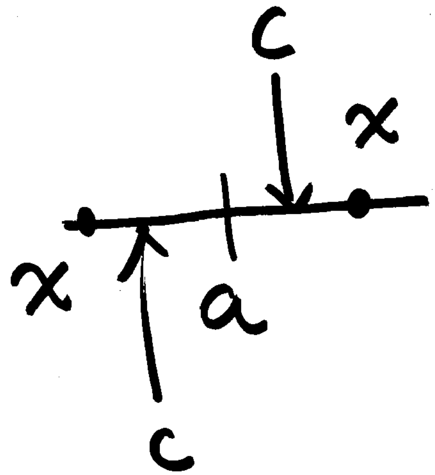


Higher order approximations :

assume that $f^{(n+1)}(a)$ exists for some $n \in \mathbb{N}$

Taylor polynomial

$$f(x) \approx f(a) + (\underbrace{x-a}_{h}) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$



$x < c < a$ or

$a < c < x$

Generalization of Taylor's Theorem

Suppose $f(x,y)$ has continuous nth partial derivatives, and $(n+1)$st partial derivatives, then

$$f(x_0+h, y_0+k) = f(x_0,y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0,y_0)$$
$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0) + \cdots$$
$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0,y_0) + \underbrace{R_n}_{\text{remainder}}$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x_0,y_0) = h\frac{\partial f(x_0,y_0)}{\partial x} + k\frac{\partial f(x_0,y_0)}{\partial y}$$

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0,y_0) =$$

$$h^2 f_{xx}(x_0,y_0) + 2hk\, f_{xy}(x_0,y_0) + k^2 f_{yy}(x_0,y_0)$$

Use binomial Thm to expand $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n$

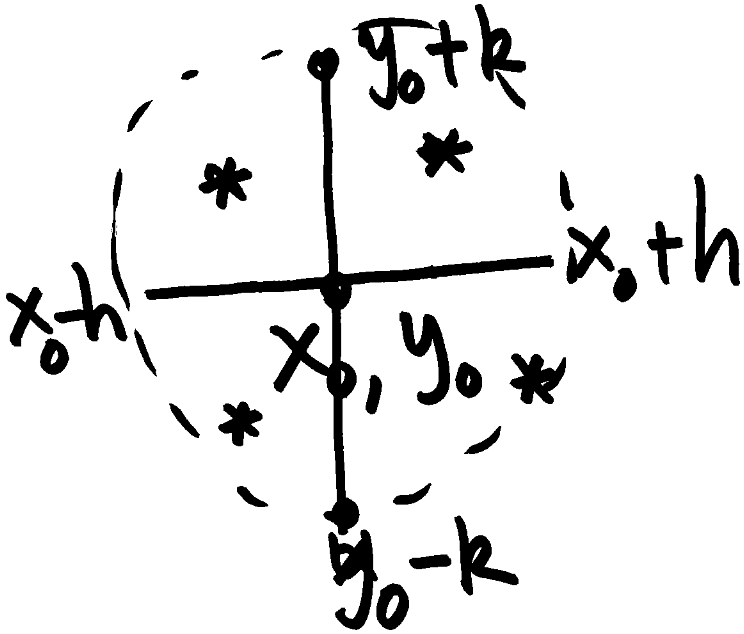
$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x_0,y_0) =$$

$$h^3 f_{xxx}(x_0,y_0) + 3h^2k\, f_{xxy}(x_0,y_0) +$$

$$3hk^2 f_{xyy}(x_0,y_0) + k^3 f_{yyy}(x_0,y_0)$$

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,\ y_0+\theta k)$$

$0 < \theta < 1$



$\sqrt{h^2+k^2}$

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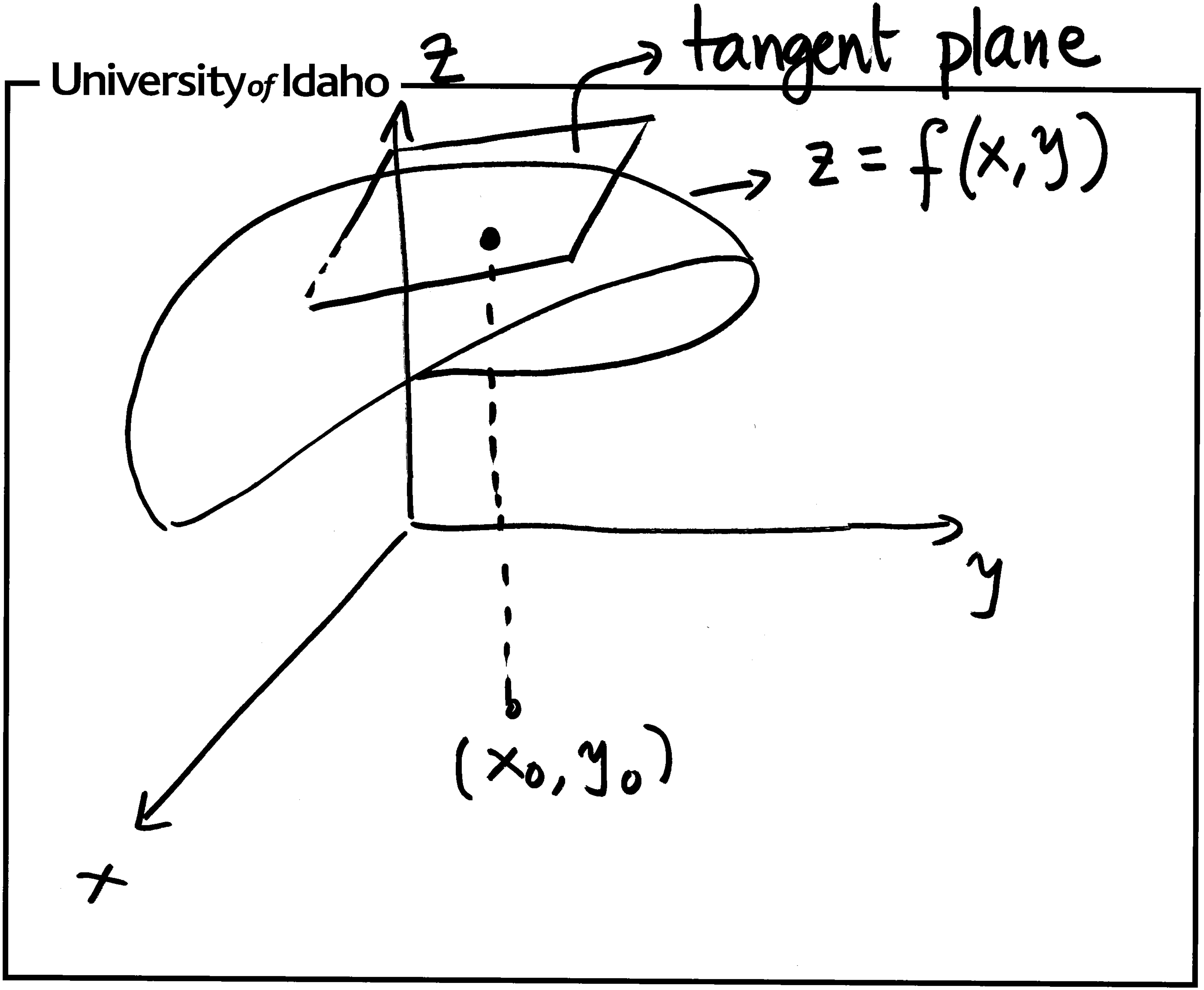

The linear approximation in this Case is

$$f(\underbrace{x_0+h}_{x}, \underbrace{y_0+k}_{y}) \approx f(x_0,y_0)+hf_x(x_0,y_0)+ k\,f_y(x_0,y_0)$$

$$z=f(x,y) \approx f(x_0,y_0)+(x-x_0)f_x(x_0,y_0) + (y-y_0)f_y(x_0,y_0)$$

This is a plane called the tangent plane of f at (x_0,y_0).

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Example

$f(x,y) = \sin(x - y - y^2)$

Find the tangent plane at $(0,0)$:

$f_x = \cos(x-y-y^2)$, $f_x(0,0) = 1$

$f_y = (-1-2y)\cos(x-y-y^2)$, $f_y(0,0) = -1$

$f(0,0) = 0$

The equation of the tangent plane is

$Z = f(0,0) + x f_x(0,0) + y f_y(0,0) = x - y$

$Z = x - y$ or $x - y - Z = 0$

At $(\pi, 0)$: $f_x(\pi, 0) = \cos \pi = -1$

$f_y(\pi, 0) = -1 \cos \pi = 1$

$f(\pi, 0) = \sin \pi = 0$

Tangent plane is

$$z = f(\pi, 0) + (x - \pi) f_x(\pi, 0) + (y - 0) f_y(\pi, 0)$$

$$= (x - \pi)(-1) + y$$

$$\Rightarrow z = \pi - x + y$$