MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 30

University of Idaho

Implicit Function Theorem

University of Idaho

Explicit $\quad y = f(x) \quad y = x^2 + \cos x$

Implicit defn. $\quad F(x,y) = 0$

Implicit: $\underbrace{\overbrace{\ln(x+y) + xy - 2}^{F} = 0}_{\text{Cannot separate } x \,\&\, y}$

Given $F(x,y) = 0$, suppose we want to find $\dfrac{dy}{dx}$

3

University of Idaho

Chain rule [Calc III]

$z = f(x,y)$ $\quad$ $x = x(t)$, $y = y(t)$

$z = f(x(t), y(t)) = z(t)$

(1) Suppose that x, y are differentiable

$$\frac{dz}{dt} = \frac{d}{dt} f(x(t), y(t))$$

$$= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

University of Idaho $z = f(x,y)$

(2) Suppose $x = x(t,s)$, $y = y(t,s)$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

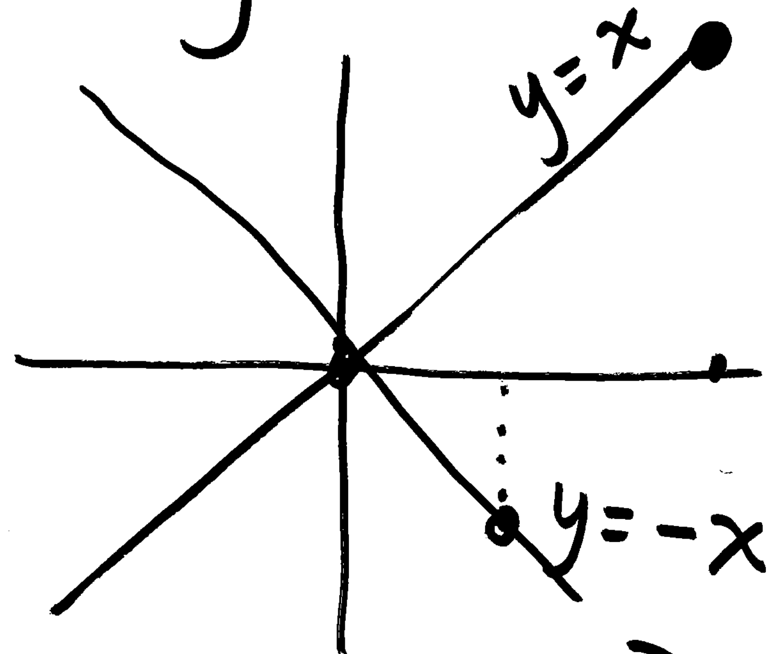
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

University of Idaho

Example

two possibilities

$$\underbrace{y^2 - x^2}_{F(x,y)} = 0 \implies y = \pm x$$



(0,0)

Given (x_0, y_0), what is f such that $f(x_0) = y_0$

$(2,-2) \longrightarrow y = -x$

$\underset{\underset{x_0}{\downarrow}}{2}, \underset{\underset{y_0}{\downarrow}}{-2}$

$(5,5) \longrightarrow y = x$

University of Idaho

$$F(x,y) = -x^2 + y^2$$

$$\underbrace{y^2 - x^2}_{F} = 0$$

$$\frac{\partial F}{\partial y} = 2y = 0 \ @ \ x=0$$

Given (x_0, y_0) one can find f s.t. $f(x_0) = y_0$ provided $x_0 \neq 0$.

$$2y\frac{dy}{dx} - 2x = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

not valid for $y \neq 0$ i.e. $x = 0$

Example

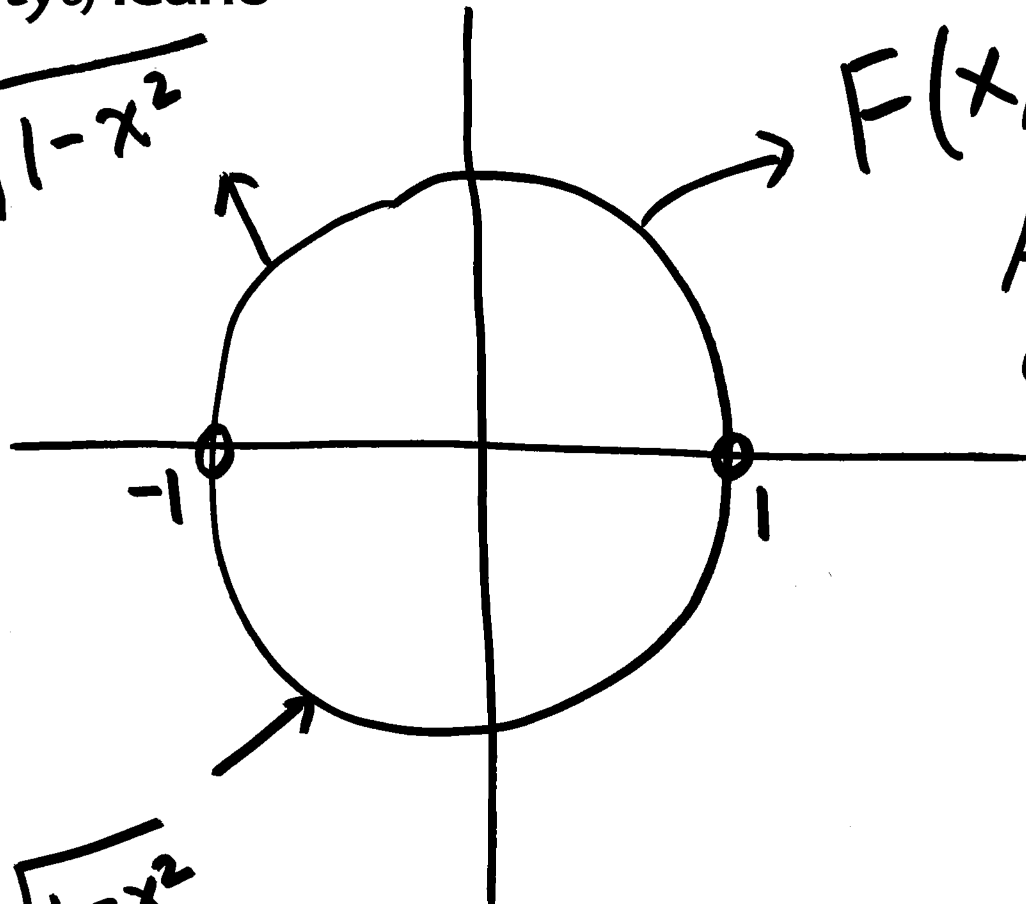
$$F(x,y) = x^2 + y^2 - 1 = 0$$

$$\Rightarrow \quad x^2 + y^2 = 1$$ unit circle

$$\Rightarrow \quad y = \pm\sqrt{1-x^2}$$ two possibilities

Given (x_0, y_0) s.t. $F(x_0, y_0) = 0$ the function f s.t. $f(x_0) = y_0$ can be found from the appropriate square root, determined by y_0 if $x_0 \neq \pm 1$

$y = \sqrt{1-x^2}$



$y = -\sqrt{1-x^2}$

$F(x,y) = 0$

At $x = \pm 1$

$\frac{dy}{dx}$ cannot exist

$y = f(x)$

Given (x₀, y₀) then $f = \sqrt{1-x_0^2}$ if $y_0 > 0$

$f = -\sqrt{1-x_0^2}$ if $y_0 < 0$

$F(x,y) = x^2 + y^2 - 1 = 0$

Exceptional points: $x_0 = \pm 1$

$f = \pm\sqrt{1-x^2}$ is not differentiable at $x_0 = \pm 1$

$$\frac{2x}{2\sqrt{1-x^2}}$$

Also, y_0 could be on either square root if $x_0 = \pm 1$.

$$\frac{\partial F}{\partial y} = 2y = \pm 2\sqrt{1-x^2} = 0$$

@ $x = \pm 1$

Implicit Function Thm

Suppose that we are given

$$F(x, y) = 0 \qquad \begin{aligned} y &= f(x) \\ x &= x \end{aligned}$$

By the chain rule:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = - \frac{\partial F/\partial x}{\partial F/\partial y}}$$

when $\partial F/\partial y \neq 0$