MATH 472

INTRODUCTION TO ANALYSIS II

SESSION no. 31

Multiple Integration

[Double Integration]

$f(x, y)$

$f : D \longrightarrow \mathbb{R}$

$D \subseteq \mathbb{R}^n$

$D \subseteq \mathbb{R}^2$

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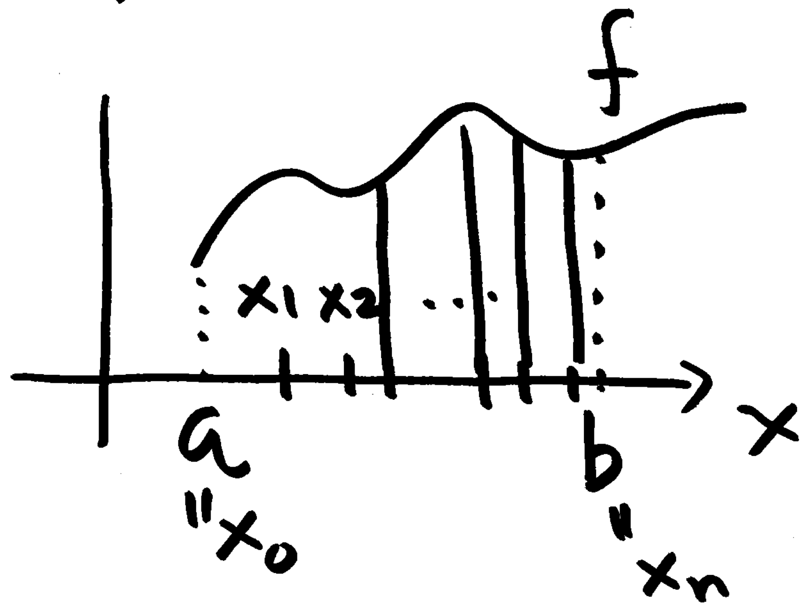
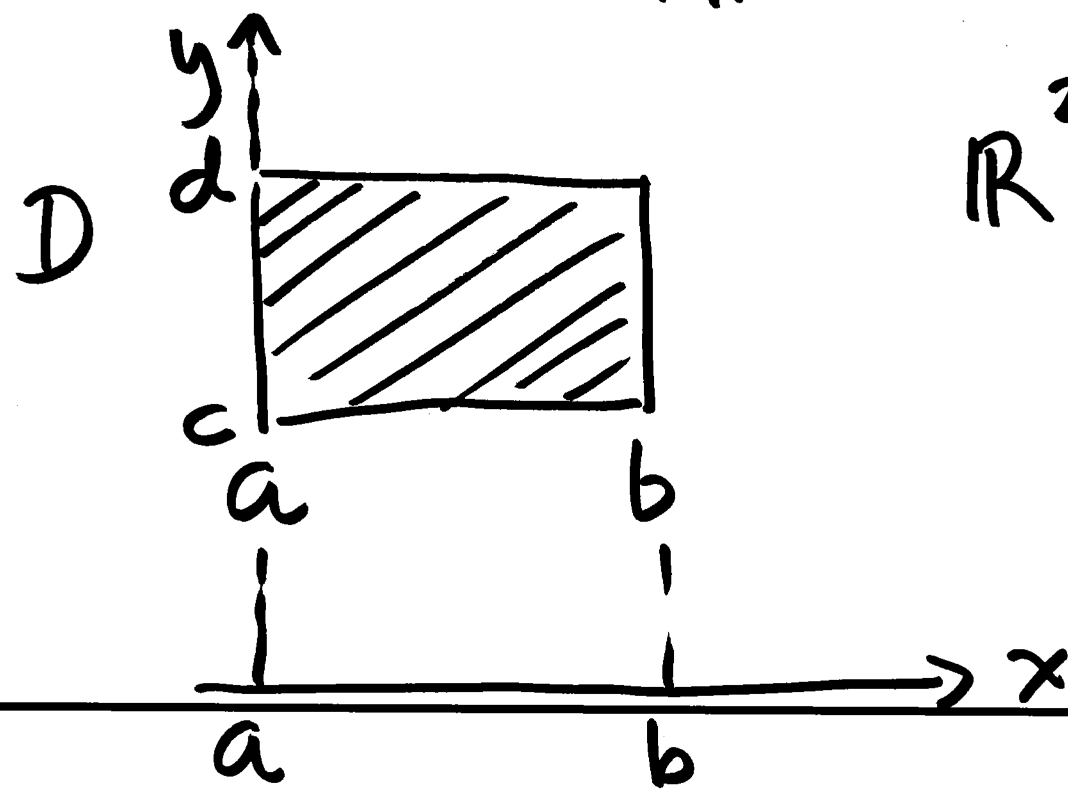


$\mathbb{R}$

$$\Delta x_i = x_i - x_{i-1}$$



$\mathbb{R}^2$

Cartesian Product

Let $A_1, A_2, \dots, A_n$ be subsets of $\mathbb{R}$.

The Cartesian product:

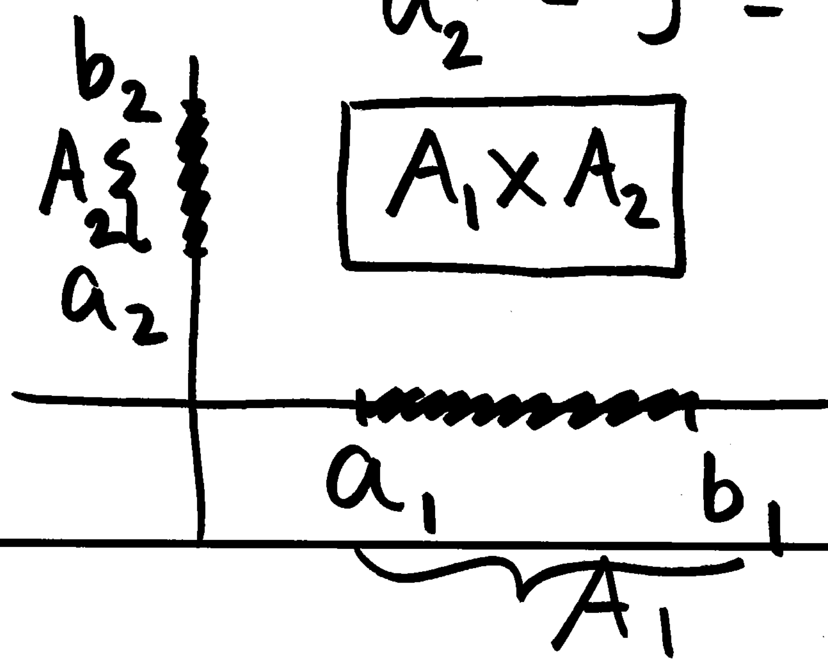
$$A_1 \times A_2 \times \cdots \times A_n$$

$$= \{ \vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 \in A_1, x_2 \in A_2, \dots, x_n \in A_n \}$$

$A_1 = [a_1, b_1] \qquad A_2 = [a_2, b_2]$

$$A_1 \times A_2 = \{(x,y) \in \mathbb{R}^2 : x \in A_1,\ y \in A_2\}$$

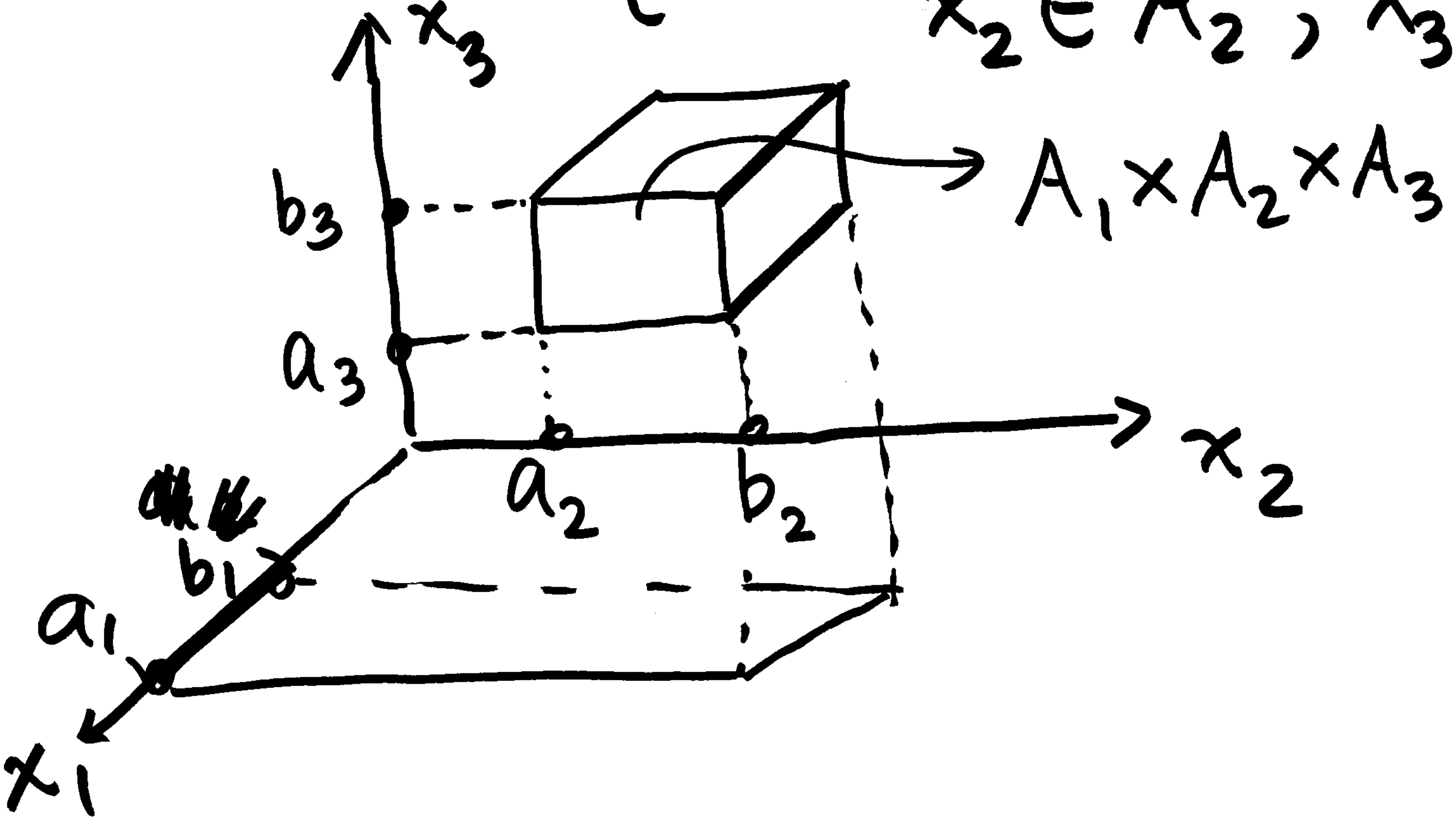
$$a_1 \le x \le b_1$$
$$a_2 \le y \le b_2$$



$A_1 \times A_2$ is the rectangle with sides given by a_1, a_2, b_1, b_2

$$A_1 = [a_1, b_1] \quad A_2 = [a_2, b_2] \quad A_3 = [a_3, b_3]$$

$$A_1 \times A_2 \times A_3 = \{ \vec{x} = (x_1, x_2, x_3) : x_1 \in A_1, x_2 \in A_2, x_3 \in A_3 \}$$



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$\subseteq \mathbb{R}$

In general, if $A_i = [a_i, b_i]$, $i = 1, 2, \ldots, n$

$A_1 \times A_2 \times \cdots \times A_n$

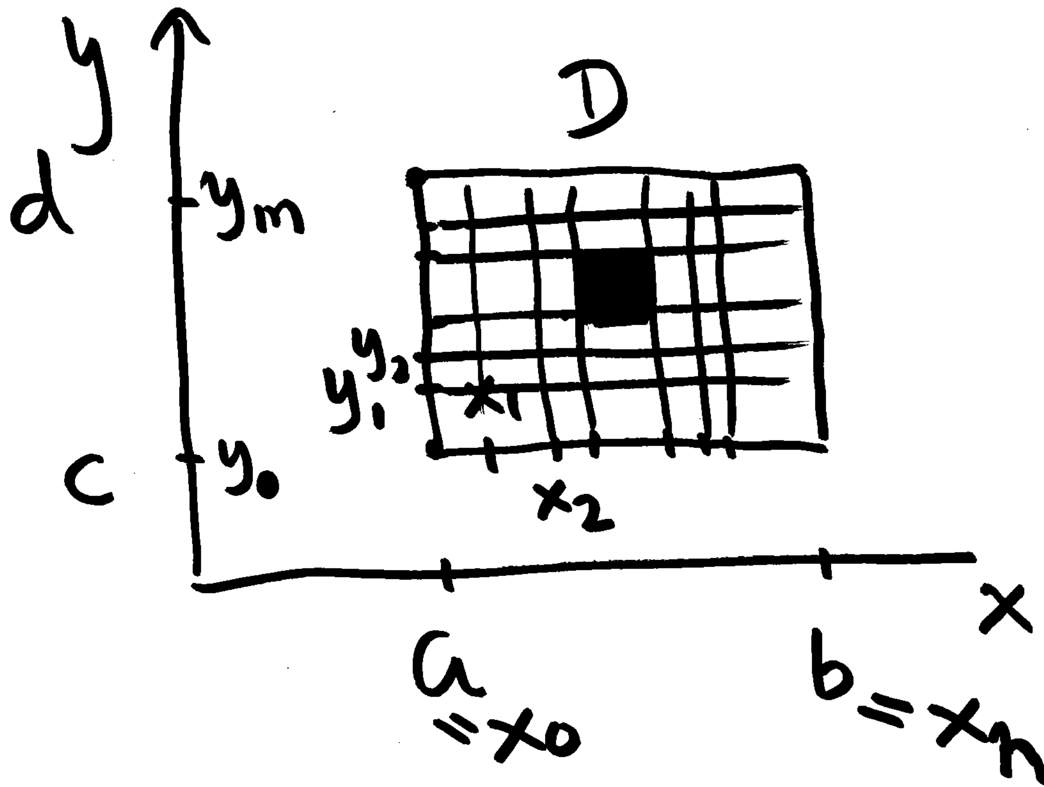
$= \{ \vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : a_i \le x_i \le b_i \, , \; i = 1, \ldots, n \}$

is called a generalized rectangle.

Double Integral

$f : D \longrightarrow \mathbb{R}$, $D \subseteq \mathbb{R}^2$

$D = [a, b] \times [c, d]$



$[x_{i-1}, x_i] \times [y_{j-1}, y_j] = R_{ij}$ (shaded)

P_1 = partition of $[a, b]$

$$P_1 = \{\underset{x_0}{\underset{\parallel}{a}}, x_1, x_2, \ldots, x_{n-1}, \overset{x_n}{\overset{\parallel}{b}}\}$$

$$= \{[x_{i-1}, x_i],\ i = 1, \ldots, n\}$$

P_2 = partition of $[c, d]$

$$= \{\underset{y_0}{\underset{\parallel}{c}}, y_1, y_2, \ldots, y_{m-1}, \underset{y_m}{\underset{\parallel}{d}}\}$$

$$= \{[y_{j-1}, y_j],\ j = 1, \ldots, m\}$$

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$$P := P_1 \times P_2$$

area of R_{ij} = $\Delta x_i \, \Delta y_j$

$$= \{R_{ij}\}_{\substack{i=1\cdots n \\ j=1\cdots m}} = \left\{ [x_{i-1}, x_i] \times [y_{j-1}, y_j] \right\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}$$

P is a grid or partition of the rectangle D.

$\Delta x_i = x_i - x_{i-1}$ $\quad$ $\Delta y_j = y_j - y_{j-1}$

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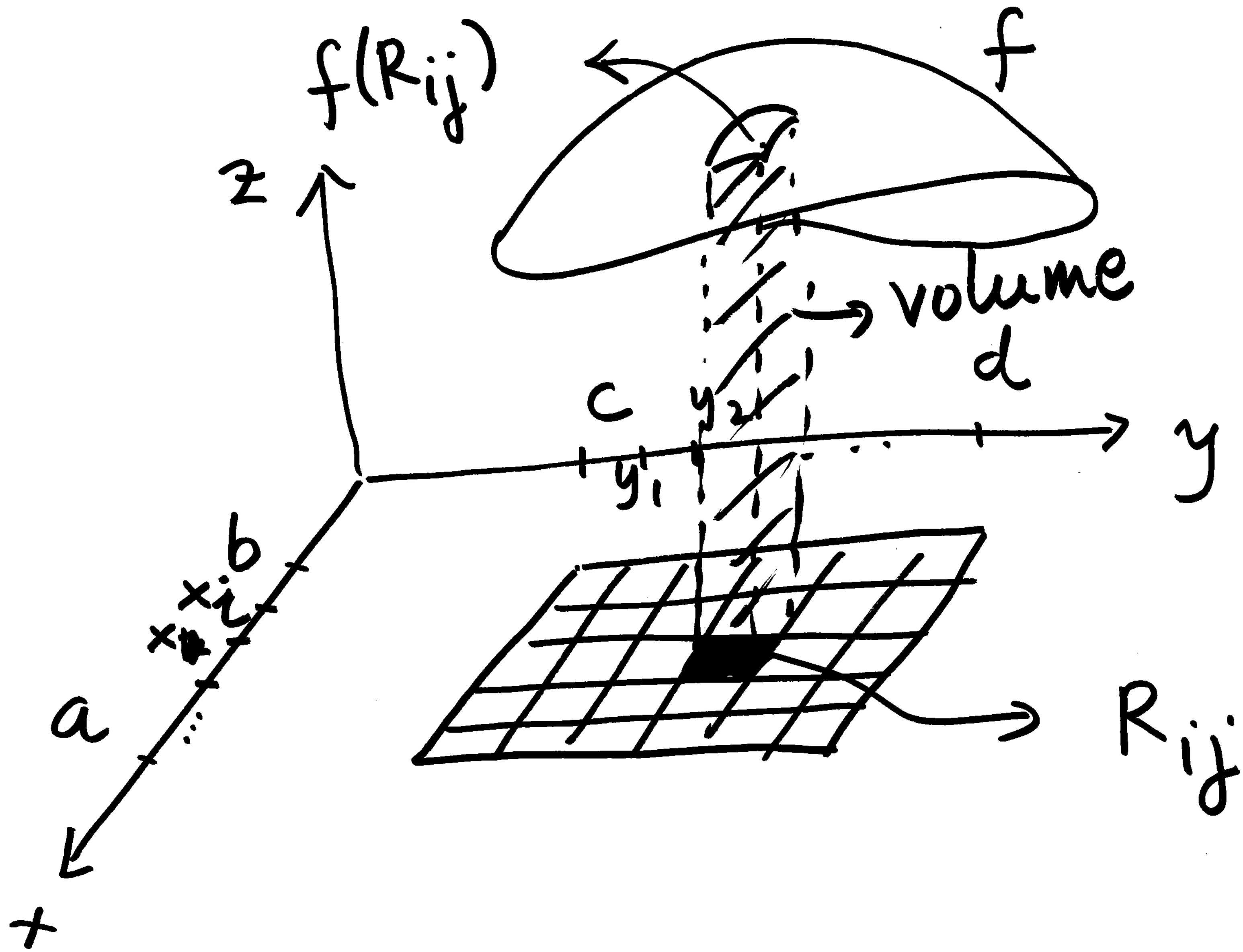
$$f : D \to \mathbb{R},\ D \subseteq \mathbb{R}^2$$



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Define:

$$M_{ij}(f) = \sup\{f(x,y) : (x,y) \in R_{ij}\}$$

$$m_{ij}(f) = \inf\{f(x,y) : (x,y) \in R_{ij}\}$$

Upper sum: $$U(P,f) = \sum_{i=1}^{n}\sum_{j=1}^{m} \overbrace{M_{ij}\,\underbrace{\text{area}(R_{ij})}_{\Delta x_i \Delta y_j}}^{\text{a volume}}$$

Lower sum: $$L(P,f) = \sum_{i=1}^{n}\sum_{j=1}^{m} \underbrace{m_{ij}\,\text{area}(R_{ij})}_{\text{a volume}}$$

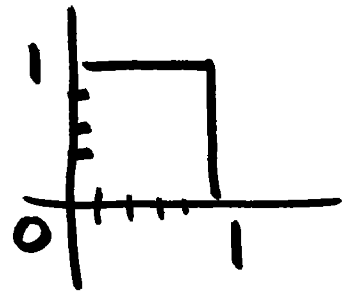
Example

$$f(x,y) = x^2 y^2 \qquad f : I \longrightarrow \mathbb{R}$$

$$I = [0,1] \times [0,1]$$

Consider the partition

$$P = \left\{ \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right] \quad 1 \le i \le n,\ 1 \le j \le n \right\}$$



$$[0,1] = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}$$

$$m_{ij}(P,f) = \frac{(i-1)^2}{n^2} \frac{(j-1)^2}{n^2}$$

$$\text{area}(R_{ij}) = \frac{1}{n^2}$$

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$$M_{ij} = \sup \text{ on } R_{ij}$$

$$= \frac{i^2}{n^2} \frac{j^2}{n^2}$$

area(R_{ij})

$$U(P,f) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{i^2 j^2}{n^4} \frac{1}{n^2}$$

$$L(P,f) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{(i-1)^2 (j-1)^2}{n^4} \frac{1}{n^2}$$